Unit 7 Progress Check: FRQ Part A

1. NO CALCULATOR IS ALLOWED FOR THIS QUESTION.

Show all of your work, even though the question may not explicitly remind you to do so. Clearly label any functions, graphs, tables, or other objects that you use. Justifications require that you give mathematical reasons, and that you verify the needed conditions under which relevant theorems, properties, definitions, or tests are applied. Your work will be scored on the correctness and completeness of your methods as well as your answers. Answers without supporting work will usually not receive credit.

Unless otherwise specified, answers (numeric or algebraic) need not be simplified. If your answer is given as a decimal approximation, it should be correct to three places after the decimal point.

Unless otherwise specified, the domain of a function f is assumed to be the set of all real numbers x for which $f(x)$ is a real number.

The population of a virus in a host can be modeled by the function A that satisfies the differential equation $\frac{dA}{dt} = t - \frac{1}{5}A$, where A is measured in millions of virus cells and t is measured in days for $5 \le t < 10$. At time $t = 5$ days, there are 10 million cells of the virus in the host.

(a) Write an equation for the line tangent to the graph of A at $t = 5$. Use the tangent line to approximate the number of virus cells in the host, in millions, at time $t = 7$ days.

Please respond on separate paper, following directions from your teacher.

(b) Show that $A(t) = -25 + 5t + 10e^{-\frac{t}{5}+1}$ satisfies the differential equation $\frac{dA}{dt} = t - \frac{1}{5}A$ with initial condition $A(5) = 10$.

Please respond on separate paper, following directions from your teacher.

(c) The host receives an antiviral medication. The amount of medication in the host is modeled by the function M that satisfies the differential equation $\frac{dM}{dt} = -\frac{1}{2}\left(\frac{M}{t+k}\right)$, where M is measured in milligrams, t is measured in days since the host received the medication, and k is a positive constant. If the amount of medication in the host is 30 milligrams at time $t = 0$ days and 15 milligrams at time $t = 3$ days, what is $M(t)$ in terms of t ?

Please respond on separate paper, following directions from your teacher.

Part A

For the second point, it is incorrect to state $A(7) = 16$ rather than $A(7) \approx 16.$

Select a point value to view scoring criteria, solutions, and/or examples to score the response.

0	1	2

The student response accurately includes both of the criteria below.

- ☐ tangent line equation
- ☐ approximation

Solution:

$A(5) = 10$

$\left.\frac{dA}{dt}\right|_{t=5} = 5 - \frac{1}{5} \cdot 10 = 3$

An equation for the line tangent to the graph of A at $t = 5$ is $y = 10 + 3(t - 5)$.

At time $t = 7$ days, there are approximately $10 + 3(7 - 5) = 16$ million virus cells in the host. $A(7) \approx 16.$

Part B

Select a point value to view scoring criteria, solutions, and/or examples and to score the response.

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0	1	2	3 ✔

The student response accurately includes all three of the criteria below.

- ☐ verification of initial condition
- ☐ $\frac{dA}{dt} = 5 - 2e^{-\frac{t}{5}+1}$
- ☐ verification that $\frac{dA}{dt} = t - \frac{1}{5}A$

Solution:

$$A(5) = -25 + 5 \cdot 5 + 10e^{-1+1} = 10$$

$$\frac{dA}{dt} = \frac{d}{dt}\left(-25 + 5t + 10e^{-\frac{t}{5}+1}\right) = 5 + 10e^{-\frac{t}{5}+1}\left(-\frac{1}{5}\right) = 5 - 2e^{-\frac{t}{5}+1}$$

$$t - \frac{1}{5}A = t - \frac{1}{5}\left(-25 + 5t + 10e^{-\frac{t}{5}+1}\right) = t + 5 - t - 2e^{-\frac{t}{5}+1} = 5 - 2e^{-\frac{t}{5}+1}$$

Part C

Zero out of 4 points earned if no separation of variables.

At most 2 out of 4 points earned [1-1-0-0] if no constant of integration.

Both antiderivatives must be correct to earn the second point.

The fourth point requires an expression for $M(t)$. The domain of $M(t)$ is included with the solution; this is not a requirement to earn the fourth point.

Select a point value to view scoring criteria, solutions, and/or examples and to score the response.

0	1	2	3	4 ✔

The student response accurately includes all four of the criteria below.

- ☐ separation of variables
- ☐ antiderivatives
- ☐ constant of integration and uses initial conditions
- ☐ $M(t)$

Solution:

$\frac{dM}{dt} = -\frac{1}{2}\left(\frac{M}{t+k}\right)$

$\frac{dM}{M} = -\frac{1}{2}\left(\frac{dt}{t+k}\right)$

$\int \frac{1}{M}\, dM = -\frac{1}{2}\int \left(\frac{1}{t+k}\right)\, dt$

$\ln|M| = -\frac{1}{2}\ln|t+k| + C$

From the context, M and t are nonnegative. It is given that $k > 0$.

$\ln M = -\frac{1}{2}\ln(t+k) + C$

$\ln M = \ln\frac{1}{\sqrt{t+k}} + C$

$M = \frac{D}{\sqrt{t+k}}$

$30 = \frac{D}{\sqrt{k}} \Rightarrow D = 30\sqrt{k}$

$15 = \frac{D}{\sqrt{3+k}} \Rightarrow D = 15\sqrt{3+k}$

$30\sqrt{k} = 15\sqrt{3+k} \Rightarrow k = 1 \text{ and } D = 30$

$M(t) = \frac{30}{\sqrt{t+1}}$

Note: This is valid for $t \geq 0$.

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2. NO CALCULATOR IS ALLOWED FOR THIS QUESTION.

Show all of your work, even though the question may not explicitly remind you to do so. Clearly label any functions, graphs, tables, or other objects that you use. Justifications require that you give mathematical reasons, and that you verify the needed conditions under which relevant theorems, properties, definitions, or tests are applied. Your work will be scored on the correctness and completeness of your methods as well as your answers. Answers without supporting work will usually not receive credit.

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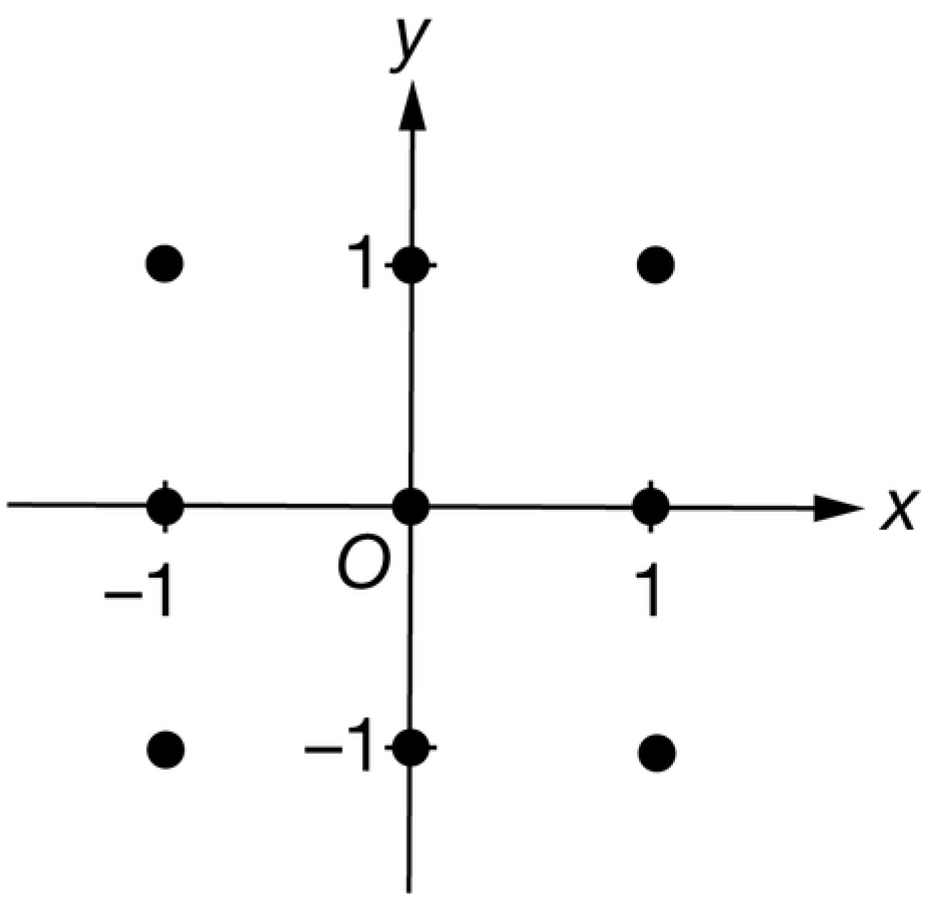
Unless otherwise specified, answers (numeric or algebraic) need not be simplified. If your answer is given as a decimal approximation, it should be correct to three places after the decimal point.

Unless otherwise specified, the domain of a function f is assumed to be the set of all real numbers x for which $f(x)$ is a real number.

Consider the differential equation $\frac{dy}{dx} = xy^4$.

(a) On the axes provided, sketch a slope field for the given differential equation at the 9 points indicated.



Please respond on separate paper, following directions from your teacher.

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(b) Find $\frac{d^2y}{dx^2}$ in terms of x and y. Determine the concavity of all solution curves for the given differential equation in Quadrant II. Give a reason for your answer.

Please respond on separate paper, following directions from your teacher.

(c) Find the particular solution $y = f(x)$ to the given differential equation with initial condition $f(4) = -1$.

Please respond on separate paper, following directions from your teacher.

Part A

The first point is earned for having all 5 correct segments of zero slope.

The second point is earned for having all 4 correct segments for the other slopes.

Select a point value to view scoring criteria, solutions, and/or examples and to score the response.

0	1	2

The student response accurately includes both of the criteria below.

- ☐ zero slopes
- ☐ other slopes

Solution:

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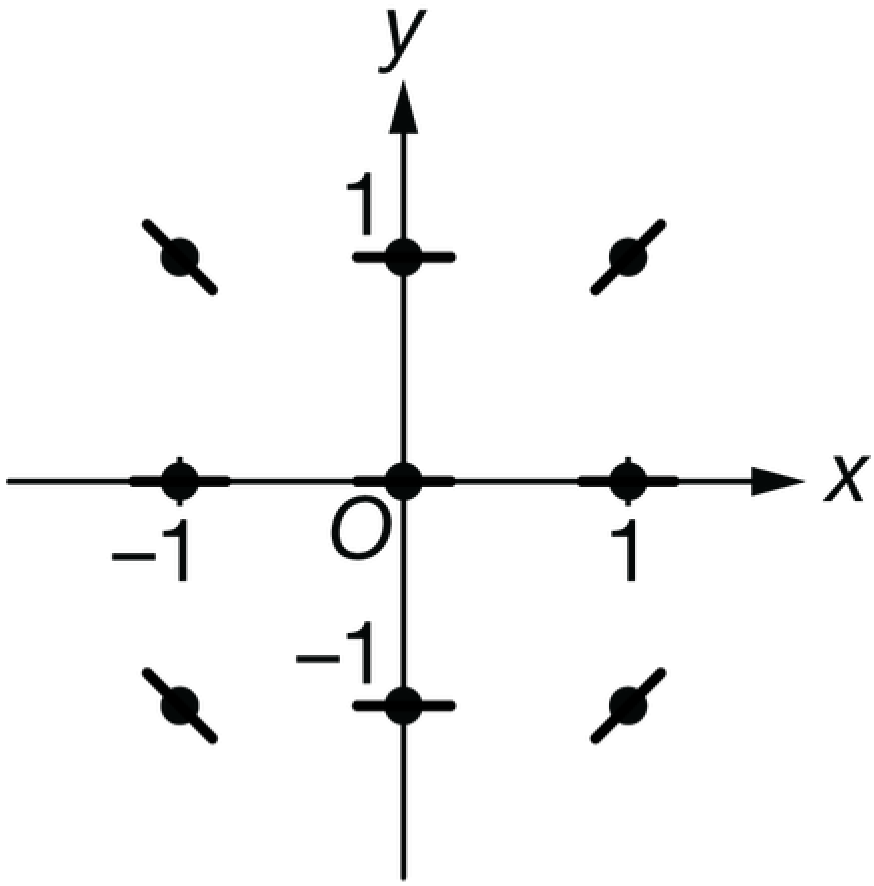

Part B

Select a point value to view scoring criteria, solutions, and/or examples and to score the response.

0	1	2

The student response accurately includes both of the criteria below.

- ☐ $\frac{d^2y}{dx^2}$
- ☐ concave up with reason

Solution:

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$\frac{d^2y}{dx^2} = 4xy^3\frac{dy}{dx} + y^4 = 4xy^3 \cdot xy^4 + y^4 = 4x^2y^7 + y^4$

In Quadrant II, $x < 0$ and $y > 0$, so $4x^2y^7 + y^4 > 0$.

Therefore, in Quadrant II all solution curves are concave up.

Part C

Zero out of 5 points earned if no separation of variables.

At most 3 out of 5 points earned [1-1-1-0-0] if no constant of integration.

The fifth point requires an expression for y. The domain is included with the solution; this is not a requirement to earn the fifth point.

Select a point value to view scoring criteria, solutions, and/or examples to score the response.

0	1	2	3	4	5 ✓

The student response accurately includes all five of the criteria below.

- ☐ separation of variables
- ☐ dy antiderivative
- ☐ dx antiderivative
- ☐ constant of integration and uses initial condition
- ☐ solves for y

Solution:

$\frac{1}{y^4}\ dy = x\ dx$

$$\int \frac{1}{y^4}\ dy = \int x\ dx$$

$-\frac{1}{3y^3} = \frac{x^2}{2} + C$

$-\frac{1}{3(-1)^3} = \frac{4^2}{2} + C \ \Rightarrow\ C = -\frac{23}{3}$

$-\frac{1}{3y^3} = \frac{x^2}{2} - \frac{23}{3} = \frac{3x^2-46}{6}$

$y^3 = \frac{2}{46-3x^2}$

$y = \sqrt[3]{\frac{2}{46-3x^2}}$

Note: This solution is valid for $-\sqrt{\frac{46}{3}} < x < \sqrt{\frac{46}{3}}$.