Unit 10 Progress Check: FRQ Part A

1. NO CALCULATOR IS ALLOWED FOR THIS QUESTION.

Show all of your work, even though the question may not explicitly remind you to do so. Clearly label any functions, graphs, tables, or other objects that you use. Justifications require that you give mathematical reasons, and that you verify the needed conditions under which relevant theorems, properties, definitions, or tests are applied. Your work will be scored on the correctness and completeness of your methods as well as your answers. Answers without supporting work will usually not receive credit.

Unless otherwise specified, answers (numeric or algebraic) need not be simplified. If your answer is given as a decimal approximation, it should be correct to three places after the decimal point.

Unless otherwise specified, the domain of a function f is assumed to be the set of all real numbers x for which $f(x)$ is a real number.

Let p be a positive constant such that the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges.

(a) Does the series $\sum_{n=1}^{\infty} \frac{1}{n^{p^2}}$ converge or diverge? Justify your answer.

Please respond on separate paper, following directions from your teacher.

(b) Does the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{p+1}}$ converge absolutely, converge conditionally, or diverge? Justify your answer.

Please respond on separate paper, following directions from your teacher.

(c) Evaluate $\int_1^{\infty} \frac{1}{x^{p+1}} dx$.

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Please respond on separate paper, following directions from your teacher.

Part A

A maximum of 1 out of 3 points may be earned for a response that indicates diverges without explicit connection to p - series.

Select a point value to view scoring criteria, solutions, and/or examples to score the response.

0	1	2	3

The student response accurately includes all three of the criteria below.

- ☐ $p < 1$
- ☐ $p^2 < 1$
- ☐ diverges with justification

Solution:

Because $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is a divergent p - series, it follows that $0 < p < 1$.

$0 < p < 1 \;\Rightarrow\; 0 < p^2 < 1$

Because $\sum_{n=1}^{\infty} \frac{1}{n^{p^2}}$ is a p - series with $0 < p^2 < 1$, the series diverges.

Part B

The third point requires that the second point is earned and that the alternating series converges by the alternating series test (i.e., there is sufficient analysis to support the conclusion of absolute convergence).

Select a point value to view scoring criteria, solutions, and/or examples and to score the response.

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0	1	2	3

The student response accurately includes all three of the criteria below.

- ☐ $p+1>1$
- ☐ $\sum_{n=1}^{\infty} \frac{1}{n^{p+1}}$ converges
- ☐ $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{p+1}}$ converges absolutely

Because $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is a divergent p - series, it follows that $0<p<1.$

$0<p<1 \Rightarrow p+1>1$

Because $\sum_{n=1}^{\infty} \frac{1}{n^{p+1}}$ is a p - series with $p+1>1,$ the series converges.

$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{p+1}}$ converges by the alternating series test.

Therefore, $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{p+1}}$ converges absolutely.

Part C

The second point must be earned to be eligible to earn the third point.

Select a point value to view scoring criteria, solutions, and/or examples and to score the response.

0	1	2	3

The student response accurately includes all three of the criteria below.

- ☐ antiderivative
- ☐ limit expression
- ☐ answer

Solution:

$$\int_1^{\infty} \frac{1}{x^{p+1}}\, dx = \lim_{b\to\infty} \int_1^{b} \frac{1}{x^{p+1}}\, dx = \lim_{b\to\infty} \left(-\tfrac{1}{p} x^{-p} \Big|_{x=1}^{x=b} \right) = \lim_{b\to\infty} \left(-\tfrac{1}{p} b^{-p} + \tfrac{1}{p} \right) = \tfrac{1}{p}$$

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2. NO CALCULATOR IS ALLOWED FOR THIS QUESTION.

Show all of your work, even though the question may not explicitly remind you to do so. Clearly label any functions, graphs, tables, or other objects that you use. Justifications require that you give mathematical reasons, and that you verify the needed conditions under which relevant theorems, properties, definitions, or tests are applied. Your work will be scored on the correctness and completeness of your methods as well as your answers. Answers without supporting work will usually not receive credit.

Unless otherwise specified, answers (numeric or algebraic) need not be simplified. If your answer is given as a decimal approximation, it should be correct to three places after the decimal point.

Unless otherwise specified, the domain of a function f is assumed to be the set of all real numbers x for which $f(x)$ is a real number.

The Taylor series for a function f about $x = 0$ is given by $\sum_{k=1}^{\infty} \frac{2^k}{3k} x^k$.

(a) Find $f^{(5)}(0)$. Show the work that leads to your answer.

Please respond on separate paper, following directions from your teacher.

(b) Use the ratio test to find the interval of convergence of the Taylor series for f about $x = 0$.

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Please respond on separate paper, following directions from your teacher.

(c) Use the second-degree Taylor polynomial for f about $x = 0$ to approximate $f\left(\frac{1}{4}\right)$.

Please respond on separate paper, following directions from your teacher.

(d) Given that $\left|f'''(x)\right| \leq \frac{128}{3}$ for $-\frac{1}{4} \leq x \leq \frac{1}{4}$, use the Lagrange error bound to show that the approximation from part (c) differs from $f\left(\frac{1}{4}\right)$ by at most $\frac{1}{9}$.

Please respond on separate paper, following directions from your teacher.

Part A

Numerical answers do not need to be simplified. A response of $\frac{f^{(5)}(0)}{5!}$ is not sufficient to earn the point.

Select a point value to view scoring criteria, solutions, and/or examples to score the response.

0	1

The student response accurately includes a correct value.

The coefficient of x^5 in the series is $\frac{f^{(5)}(0)}{5!} = \frac{2^5}{3 \cdot 5}$.

Therefore, $f^{(5)}(0) = 5! \cdot \frac{2^5}{3 \cdot 5} = 120 \cdot \frac{32}{15} = 256.$

Part B

The first point is earned for a correct ratio; the limit and absolute value are not required for the first point.

The second point requires appearance of absolute value and use of limit notation. Incorrect mathematical notation such as use of $\frac{0}{0}$ or $\frac{\infty}{\infty}$ impacts this point.

The third point may be earned with an incorrect limit evaluation provided the limit is nonzero and does not produce a trivial result for the radius of convergence. A statement such as $2|x| < 1$ is incomplete and requires evidence of "considering" the endpoints to demonstrate evidence of a "radius."

The fourth point requires substitution of both endpoints into the general term. A maximum of one computational error is allowed, provided the response demonstrates the required substitution.

The fifth point requires justification of convergence/divergence at each endpoint based on an appropriate test and/or reference to an appropriate series.

Select a point value to view scoring criteria, solutions, and/or examples and to score the response.

0	1	2	3	4	5

The student response accurately includes all five of the criteria below.

- ☐ sets up ratio
- ☐ computes limit of ratio
- ☐ radius of convergence
- ☐ consider both endpoints
- ☐ analysis and interval of convergence

Solution:

$$\lim_{k\to\infty}\left|\frac{\frac{2^{k+1}x^{k+1}}{3(k+1)}}{\frac{2^k x^k}{3k}}\right| = \lim_{k\to\infty}\frac{2k}{k+1}|x| = 2|x|$$

$$2|x| < 1 \;\Rightarrow\; |x| < \frac{1}{2}$$

Therefore, the radius of convergence of the Taylor series for f about $x = 0$ is $\frac{1}{2}$.

When $x = -\frac{1}{2}$, the series is $\sum_{k=1}^{\infty}\frac{(-1)^k}{3k}$, which is an alternating series with individual terms that decrease in absolute value to 0. This series converges by the alternating series test.

When $x = \frac{1}{2}$, the series is $\sum_{k=1}^{\infty} \frac{1}{3k}$. This series diverges by limit comparison with the harmonic series.

The interval of convergence of the Taylor series for f about $x = 0$ is $-\frac{1}{2} \leq x < \frac{1}{2}$.

Part C

Supporting work is required to earn the point. Numerical answers do not need to be simplified.

Select a point value to view scoring criteria, solutions, and/or examples and to score the response.

0	1 ✓

The student response accurately includes a correct approximation.

Solution:

The second-degree Taylor polynomial for f about $x = 0$ is $P_2(x) = \frac{2^1}{3 \cdot 1}x + \frac{2^2}{3 \cdot 2}x^2 = \frac{2}{3}x + \frac{2}{3}x^2$.

$$f\left(\frac{1}{4}\right) \approx P_2\left(\frac{1}{4}\right) = \frac{2}{3}\left(\frac{1}{4}\right) + \frac{2}{3}\left(\frac{1}{4}\right)^2 = \frac{5}{24}$$

Part D

Both points may be earned with $\text{Error} \leq \frac{\frac{128}{3}}{6 \cdot 4^3} = \frac{1}{9}$ or equivalent. The first point requires $\text{Error} \leq \frac{\max\left|f'''(x)\right|}{3!} \cdot \left(\frac{1}{4}\right)^3$ or equivalent. The second point requires an explicit connection to the error and indicates the numeric value is clearly less than or equal to $\frac{1}{9}$.

The response is not eligible for either point with use of the alternating series error bound.

Select a point value to view scoring criteria, solutions, and/or examples and to score the response.

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0	1	2

The student response accurately includes both of the criteria below.

- ☐ form of the error bound
- ☐ analysis

Solution:

$$\text{Error} \le \frac{\max\limits_{0 \le x \le \frac{1}{4}} \left| f'''(x) \right|}{3!} \cdot \left(\frac{1}{4}\right)^3 \le \frac{\frac{128}{3}}{6 \cdot 4^3} = \frac{1}{9}$$