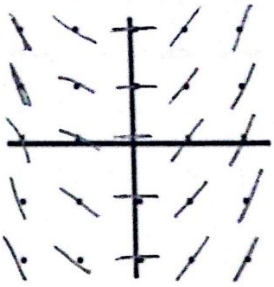
AP Calculus AB Unit 6B:
Euler's Method, Growth and Decay, and Logistic Functions

Name: KEY

Warm-Up:

Given dy/dx = x

1) Draw a slope field.



2) Find the general solution of the DiffEQ. $\int dy = \int x\,dx$

$y = \frac{x^2}{2} + C$

3) Find the particular solution if y(2) = 1.

$1 = \frac{2^2}{2} + C \quad 1 = 2 + C \quad C = -1$

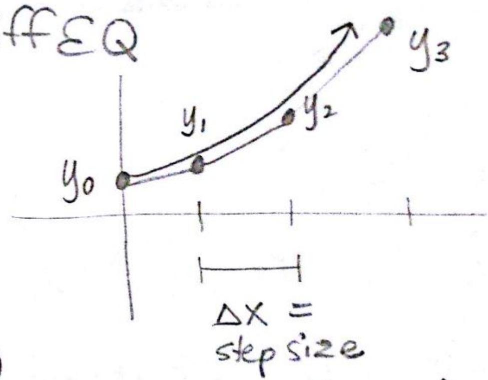
$y = \frac{x^2}{2} - 1$

4) Find y(8) using your solution from #3.

$y(8) = \frac{64}{2} - 1 = 31$

Euler's Method:

iterative (recursive) process which gives a numerical method to approximate the particular solution to a DiffEQ



$y - y_1 = m(x - x_1)$

$y = m(x - x_1) + y_1$

$y_{new} = \frac{dy}{dx}(\Delta x) + y_{old}$

1) Given $\frac{dy}{dx} = \frac{x}{2} + \frac{y}{5}$ and f (2) = 0. Estimate f(3) if $\Delta x = 0.5$.

x	y	dy/dx
2	0	1 + 0 = 1
2.5	.5	1.35
3	1.175	

$y = .5(1) + 0$

$y = .5(1.35) + .5$

2) Given $\frac{dy}{dx} = x + y$ and f (0) = 0. Estimate y if x = 0.5 using step size of 0.1.

x	y	dy/dx
0	0	0
0.1	0	0.1
0.2	.01	.21
0.3	.031	.331
0.4	.0641	.4641
0.5	.11051	

$y = .1(0) + 0$

$y = .1(0.1) + 0$

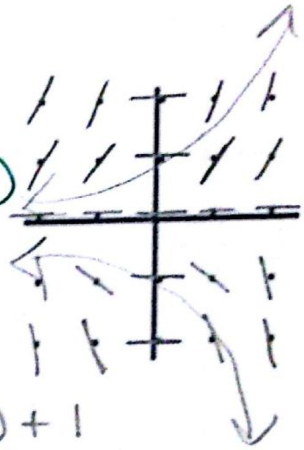
$y = .1(.21) + .01$

$y = .1(.331) + .031$

$y = .1(.4641) + .0641$

Free Response Practice:

1) Consider the DiffEQ: $\frac{dy}{dx} = x^2 y$. a) Draw a slope field for the DiffEQ. (1)



b) Let y = f(x) be the particular solution to the given DiffEQ with initial condition f (0) = 1. Use Euler's Method starting at x = 0, with a step of 0.1, to approximate f (0.2). Show the work that leads to your answer.

x	y	dy/dx
0	1	0
0.1	1 (1)	.01 (1)
0.2	1.001 (1)	

$f(0.2) \approx 1.001$

$y = .1(0) + 1$

$y = .1(.01) + 1$

c) Find the particular solution y = f(x) to the given DiffEQ with the initial condition f (0) = 1. Use your solution to find f(0.2).

(1) $\int \frac{1}{y}\,dy = \int x^2\,dx$

(1) $\ln|y| = \frac{x^3}{3} + C$ (1)

$y = \pm e^{\frac{x^3}{3} + C}$

$y = \pm e^{\frac{x^3}{3}} \cdot e^C$

$y = Ce^{x^3/3}$

$1 = Ce^0$ (1)

$C = 1$

$y = e^{x^3/3}$

$y = e^{(0.2)^3/3}$ (1)

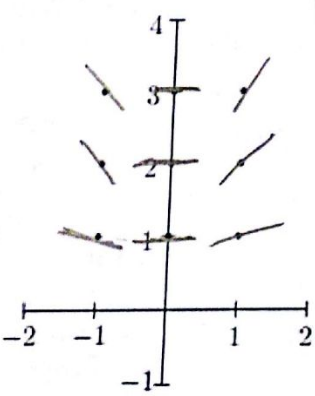
≈ 1.003

2) Consider the differential equation given by $\frac{dy}{dx} = \frac{xy}{2}$.

(a) On the graph below, sketch a slope field for the given differential equation at the nine points indicated.

x	y	dy/dx
0	3	0
0.1	3	.15
0.2	3.015	

3



1

$y = 0.1(0) + 3$

$y = 0.1(.15) + 3$

(b) Let $y = f(x)$ be the particular solution to the given differential equation with the initial condition $f(0) = 3$. Use Euler's method starting at $x = 0$, with a step size of 0.1, to approximate $f(0.2)$. Show the work that leads to your answer.

$f(0.2) \approx 3.015$

$y = e^{x^2/4} \cdot e^{C/2}$

$y = Ce^{x^2/4}$

$y(0.2) = 3.030$

(c) Find the particular solution $y = f(x)$ to the given differential equation with the initial condition $f(0) = 3$. Use your solution to find $f(0.2)$.

$2\,dy = (xy)\,dx$

$2 \cdot \int \frac{1}{y}\,dy = \int x\,dx$

$2\ln|y| = \frac{x^2}{2} + C$

$\ln|y| = \frac{x^2}{4} + \frac{C}{2}$

$3 = Ce^0$

$C = 3$

$y = 3e^{x^2/4}$

Growth and Decay:

If rate of change is ***proportional*** to amount present, change can be modeled by: * $\frac{dy}{dt} = Ky$

If rate of change is ***inversely proportional*** to amount present, change can be modeled by: $\frac{dy}{dt} = \frac{K}{y}$

Examples: * $\int \frac{1}{y}\,dy = \int K\,dt$ $\ln y = Kt + C \longrightarrow y = Ce^{Kt}$

1) The rate of change of y is proportional to y. When t = 0, y = 2. When t = 2, y = 4. What is the value of y when t = 3?

① $y = Ce^{Kt}$, $2 = Ce^0$, $C = 2$ ② $y = 2e^{Kt}$, $4 = 2e^{2K}$, $2 = e^{2K}$, $\ln 2 = 2K$, $K = \frac{\ln 2}{2}$ ③ $y = 2e^{\ln 2/2 \cdot t}$, $y = 2e^{\ln 2/2 \cdot 3} = 5.657$

2) The rate of change of P is proportional to P. When t = 0, P = 5000 and when t = 1, P = 4750. Find P when t = 5.

① $P = Ce^{Kt}$, $5000 = Ce^0$, $C = 5000$ ② $P = 5000e^{Kt}$, $4750 = 5000e^K$, $.95 = e^K$, $\ln .95 = K$ ③ $P = 5000e^{(\ln .95)t}$, $P = 5000e^{(\ln .95)5} = 3868.905$

3) The rate of decay of a radioactive substance is proportional to the amount of substance present. Four years ago there were 12 grams of the substance. Now there are 8 grams. How many grams will there be 8 years from now?

$t=0, g=12$; $t=4, g=8$; $t=12, g=?$

A) 0 B) 8/3 C) 32/9 D) 81/16 E) 16/3

$g = Ce^{Kt}$, $12 = Ce^0$, $C = 12$ $g = 12e^{Kt}$, $8 = 12e^{4K}$, $\frac{2}{3} = e^{4K}$, $\ln\frac{2}{3} = 4K$, $K = \frac{\ln\frac{2}{3}}{4}$, $g = 12e^{\left(\frac{\ln 2/3}{4}\right)t}$, $g = 12e^{\left(\frac{\ln\frac{2}{3}}{4}\right)12}$, $g = 12e^{3\ln\frac{2}{3}}$, $g = 12e^{\ln\frac{8}{27}}$, $g = 12 \cdot \frac{8}{27} = \frac{32}{9}$

4) Population y grows according to the equation dy/dt = ky, where k is a constant and t is measured in years. If the population doubles every 8 years, which of the following could be the value of k?

$t=0, y=10$; $t=8, y=20$

a) 0.087 b) 0.349 c) 0.799 d) 1.071 e) 1.152

$y = Ce^{Kt}$, $10 = Ce^0$, $C = 10$ $20 = 10e^{8K}$, $2 = e^{8K}$, $\ln 2 = 8K$ $K = \frac{\ln 2}{8}$, $K = 0.087$

Free Response Practice:

1) Oil is being pumped continuously from a certain oil well at a rate proportional to the amount of oil left in the well; that is dy/dt = ky, where y is the amount of oil left in the well at any time t. Initially there were 1,000,000 gallons of oil in the well, and 6 years later there were 500,000 gallons remaining. It will no longer be profitable to pump oil when there are fewer than 50,000 gallons remaining.

$t = 0, y = 1{,}000{,}000$; $t = 6, y = 500{,}000$

a) Write an equation for y, the amount of oil remaining in the well at any time t.

$y = Ce^{Kt}$, $1{,}000{,}000 = Ce^0$, $C = 1{,}000{,}000$ $500{,}000 = 1{,}000{,}000e^{6K}$, $\frac{1}{2} = e^{6K}$, $\ln\frac{1}{2} = 6K$ $K = \frac{\ln\frac{1}{2}}{6}$, $y = 10^6 \cdot e^{\left(\frac{\ln\frac{1}{2}}{6}\right)t}$

b) At what rate is the amount of oil in the well decreasing when there are 600,000 gallons of oil remaining?

$\frac{dy}{dt} = \frac{\ln(\frac{1}{2})}{6}(600{,}000) = -100{,}000\ln 2$

-69314.72 $\ln 1 - \ln 2 = 0 - \ln 2 = -\ln 2$

c) In order not to lose money, at what time, t, should oil no longer be pumped from the well?

$5 \times 10^4 = 10^6 e^{\left(\frac{\ln\frac{1}{2}}{6}\right)t}$, $\frac{1}{20} = e^{\left(-\frac{\ln 2}{6}\right)t}$ $\ln\frac{1}{20} = \left(-\frac{\ln 2}{6}\right)t$, $\frac{\ln\frac{1}{20}}{\left(-\frac{\ln 2}{6}\right)}$ $\to \ln\frac{1}{20} \cdot \frac{-6}{\ln 2}$, $\cancel{\ln 1} - \ln 20$, $\frac{-\ln 20 \cdot -6}{\ln 2}$ $t \geq 6\ln 20/\ln 2$

$$y = \frac{C}{1+a\cdot e^{-Kx}}$$ C = limit to growth

Logistic Functions:

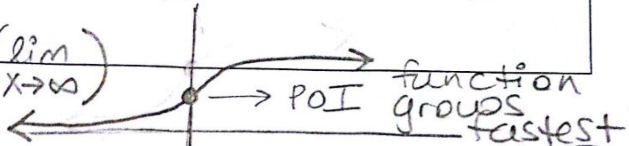
Logistic Growth Model (Pre-Calculus)	Logistic Growth Model (Calculus)
$P = \frac{M}{1+Ae^{-Kt}}$ $\longrightarrow$	$\frac{dP}{dt} = KP\left(1-\frac{P}{M}\right)$
$P = \frac{M}{1+Ae^{-(MK)t}}$ $\longrightarrow$	$\frac{dP}{dt} = KP(M-P)$

Examples:

M = carrying capacity $\left(\lim_{x\to\infty}\right)$

K = growth constant

A = constant



1) Ten grizzly bears were introduced to a national park 10 years ago. There are 23 bears in the park at the present time. The park can support a maximum of 100 bears. Assuming a logistic growth model, when will DA BEAR population reach 50? 75? 100?

$P = \frac{M}{1+Ae^{-Kt}}$ $M = 100$, $t=0, P=10$, $t=10, P=23$

$10 = \frac{100}{1+Ae^{0}}$ $10 + 10A = 100$, $A = 9$

$10 = \frac{100}{1+A}$ $P = \frac{100}{1+9e^{-Kt}}$

$23 = \frac{100}{1+9e^{-10K}}$ $K = 0.0988913$

$1+9e^{-10K} = \frac{100}{23}$

$9e^{-10K} = \frac{77}{23}$

$e^{-10K} = 0.371981$

$-10K = -0.988913$

half of carry capacity

$P = \frac{100}{1+9e^{-.099t}}$

2) The growth rate of a population P of bears in a newly established wildlife preserve is modeled by the differential dP/dt = 0.008P(100 – P), where t is measured in years.

a) What is the carrying capacity?

100

$100 = \frac{100}{1+9e^{-0.99t}}$

$9e^{-0.99t} = 0$

$e^{-0.99t} = \cancel{\ln 0}$

graph:

$y = 50$ 22 yrs

$y = 75$ 33 yrs

$y = 100$ never!

b) What is the bear population when the population is growing the fastest?

POI $\overset{?}{\nabla}$ 0 $\frac{dP}{dt} = 0.008P(100-P)$

$\frac{dP}{dt} = .8P - .008P^2$

$\frac{d^2P}{dt^2} = .8 - .016P$

$0 = .8 - .016P$

$-.8 = -.016P$

$P = 50$

c) What is the rate of change of the population when it is growing the fastest?

$\frac{dP}{dt} = 0.008(50)(100-50)$

$= 20$

*

$.008P(100-P) \rightarrow .8P\left(1-\frac{P}{100}\right)$

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