Practice Free Response

For Integration Test

1976 AB6

(a) Given $5x^3 + 40 = \int_c^x f(t)\,dt$.

(i) Find $f(x)$.

(ii) Find the value of c.

(b) If $F(x) = \int_x^3 \sqrt{1+t^{16}}\,dt$, find $F'(x)$.

1976 AB6
Solution

(a) (i) Take the derivative of both sides of $5x^3 + 40 = \int_c^x f(t)\,dt$ to get

$15x^2 + 0 = f(x)$. Thus $f(x) = 15x^2$.

(ii) Method 1 (using the result from (i)):

$$5x^3 + 40 = \int_c^x 15t^2\,dt = 5t^3\Big|_c^x = 5x^3 - 5c^3$$

$$5c^3 = -40$$

$$c = -2$$

Method 2 (using the condition given in the stem with $x = c$):

$5c^3 + 40 = \int_0^0 f(t)dt = 0$ and so $c = -2$.

(b) $F(x) = \int_x^3 \sqrt{1+t^{16}}\,dt = -\int_3^x \sqrt{1+t^{16}}\,dt$

$$F'(x) = \frac{d}{dx}\left(-\int_3^x \sqrt{1+t^{16}}\,dt\right) = -\sqrt{1+x^{16}}$$

1977 BC7

Let $F(x)=\int_0^x \frac{1}{1+t^4}\,dt$ for all real numbers x.

(a) Find $F(0)$.

(b) Find $F'(1)$.

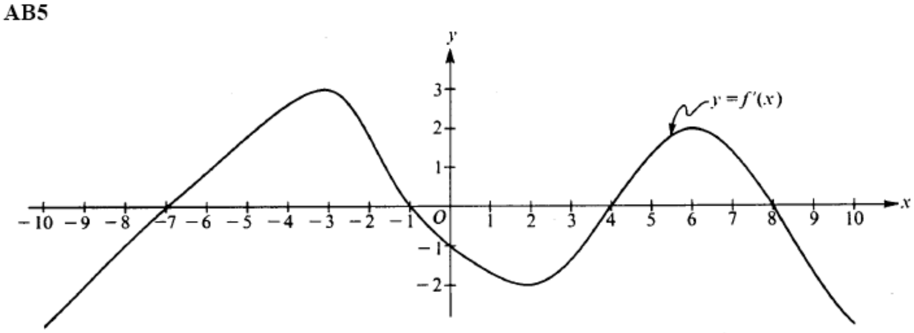
1977 BC7
Solution

(a) $F(0)=\int_0^0 \frac{1}{1+t^4}=0$

(b) $F'(x)=\frac{1}{1+x^4}$

$F'(1)=\frac{1}{2}$

1989 AB5



Note: This is the graph of the derivative of f, not the graph of f.

The figure above shows the graph of f', the derivative of a function f. The domain of f is the set of all real numbers x such that $-10\le x\le 10$.

(a) For what values of x does the graph of f have a horizontal tangent?

(b) For what values of x in the interval $(-10,10)$ does f have a relative maximum? Justify your answer.

(c) For value of x is the graph of f concave downward?

1989 AB5
Solution

(a) horizontal tangent $\Leftrightarrow f'(x)=0$
$x=-7,-1,4,8$

(b) Relative maxima at $x=-1,\ 8$ because f' changes from positive to negative at these points

(c) f concave downward $\Leftrightarrow f'$ decreasing
$(-3,2),\ (6,10)$

1991 AB1

Let f be the function that is defined for all real numbers x and that has the following properties.

(i) $f''(x)=24x-18$

(ii) $f'(1)=-6$

(iii) $f(2)=0$

(a) Find each x such that the line tangent to the graph of f at $(x, f(x))$ is horizontal.

(b) Write an expression for $f(x)$.

(c) Find the average value of f on the interval $1 \le x \le 3$.

1991 AB1
Solution

(a) $f'(x) = 12x^2 - 18x + C$

$f'(1) = -6 = 12 - 18 + C$

Therefore $C = 0$

$f'(x) = 6x(2x - 3) = 0$

$x = 0, \frac{3}{2}$

(b) $f(x) = 4x^3 - 9x^2 + C$

$f(2) = 0 = 32 - 36 + C$

Therefore $C = 4$

$f(x) = 4x^3 - 9x^2 + 4$

(c) $\frac{1}{3-1}\int_1^3 4x^3 - 9x^2 + 4\,dx$

$= \frac{1}{2}\left[x^4 - 3x^3 + 4x\right]\Big|_1^3$

$= \frac{1}{2}\left[(81 - 81 + 12) - (1 - 3 + 4)\right]$

$= 5$

1994 AB 1

Let f be the function given by $f(x) = 3x^4 + x^3 - 21x^2$.

(a) Write an equation of the line tangent to the graph of f at the point $(2, -28)$.

(b) Find the absolute minimum value of f. Show the analysis that leads to your conclusion.

(c) Find the x-coordinate of each point of inflection on the graph of f. Show the analysis that leads to your conclusion.

1994 AB 1

(a) $f'(x) = 12x^3 + 3x^2 - 42x$

$f'(2) = 24$

$y + 28 = 24(x - 2)$

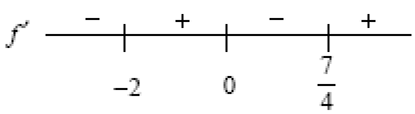
or $y = 24x - 76$

(b) $12x^3 + 3x^2 - 42x = 0$

$3x(4x^2 + x - 14) = 0$

$3x(4x - 7)(x + 2) = 0$

$x = 0, x = \frac{7}{4}, x = -2$



min must be at -2 or $\frac{7}{4}$.

$f(-2) = -44 \quad f\left(\frac{7}{4}\right) = -30.816$

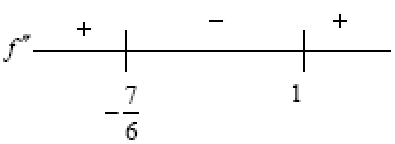
Absolute min is -44

(c) $f''(x) = 36x^2 + 6x - 42$

$= 6(6x^2 + x - 7)$

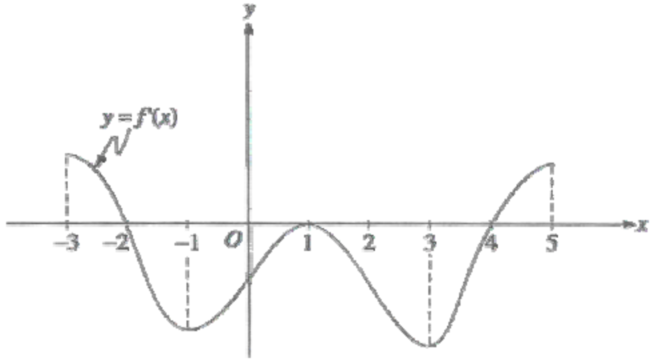
$= 6(6x + 7)(x - 1)$

Zeros at $x = -\frac{7}{6}, \quad x = 1$

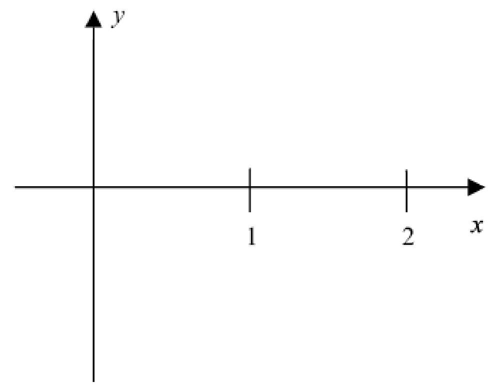


The x coordinates of the points of inflection are $x = -\frac{7}{6}$ and $x = 1$

1996 AB1



Note: This is the graph of the derivative of f, not the graph of f.



The figure above shows the graph of f', the derivative of a function f. The domain of f is the set of all real numbers x such that $-3 < x < 5$.

(a) For what values of x does f have a relative maximum? Why?

(b) For what values of x does f have a relative minimum? Why?

(c) On what intervals is the graph of f concave upward? Use f' to justify your answer.

(d) Suppose that $f(1) = 0$. In the xy-plane provided, draw a sketch that shows the general shape of the graph of the function f on the open interval $0 < x < 2$.

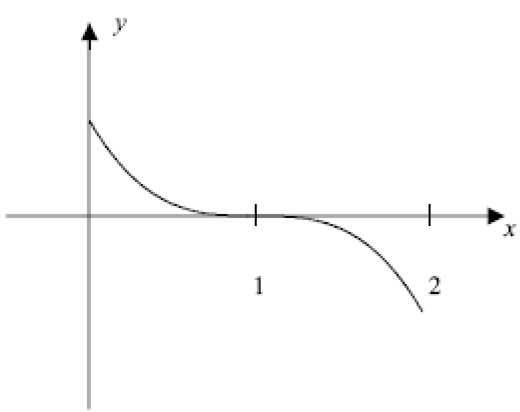
1996 AB1
Solution

(a) $x = -2$
$f'(x)$ changes from positive to negative at $x = -2$
or
f is increasing to the left of $x = -2$ and decreasing to the right of $x = -2$

(b) $x = 4$
$f'(x)$ changes from negative to positive at $x = 4$
or
f is decreasing to the left of $x = 4$ and increasing to the right of $x = 4$

(c) $(-1,1)$ and $(3,5)$
f' is increasing on these intervals.

(d)



1970 AB3/BC2

Consider the function f given by $f(x) = x^{\frac{4}{3}} + 4x^{\frac{1}{3}}$ on the interval $-8 \le x \le 8$.

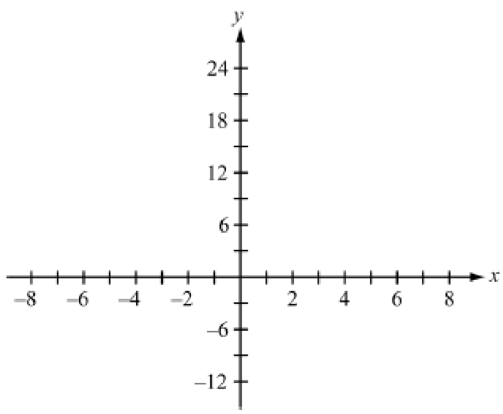
(a) Find the coordinates of all points at which the tangent to the curve is a horizontal line.

(b) Find the coordinates of all points at which the tangent to the curve is a vertical line.

(c) Find the coordinates of all points at which the absolute maximum and absolute minimum occur.

(d) For what values of x is this function concave down?

(e) On the axes provided, sketch the graph of the function on this interval.



1970 AB3/BC2
Solution

(a) $f(x) = x^{4/3} + 4x^{1/3} = x^{1/3}(x+4)$

$$f'(x) = \frac{4}{3}x^{1/3} + \frac{4}{3}x^{-2/3} = \frac{4}{3}\left(\frac{x+1}{x^{2/3}}\right)$$

$f'(x) = 0$ at $x = -1$. There is a horizontal tangent at $(-1,-3)$.

(b) There is a vertical tangent at $(0,0)$.

(c) The absolute maximum and absolute minimum must occur at a critical point or an endpoint. The candidates are

$(-8,8)$, $(-1,-3)$, $(0,0)$, and $(8,24)$

So the absolute maximum is at $(8,24)$ and the absolute minimum is at $(-1,-3)$.

(d) $$f''(x) = \frac{4}{9}x^{-2/3} - \frac{8}{9}x^{-5/3} = \frac{4}{9}\left(\frac{x-2}{x^{5/3}}\right)$$

The graph is concave down for $0 < x < 2$.

(e)