Unit 4 Progress Check: FRQ Part B

1. NO CALCULATOR IS ALLOWED FOR THIS QUESTION.

Show all of your work, even though the question may not explicitly remind you to do so. Clearly label any functions, graphs, tables, or other objects that you use. Justifications require that you give mathematical reasons, and that you verify the needed conditions under which relevant theorems, properties, definitions, or tests are applied. Your work will be scored on the correctness and completeness of your methods as well as your answers. Answers without supporting work will usually not receive credit.

Unless otherwise specified, answers (numeric or algebraic) need not be simplified. If your answer is given as a decimal approximation, it should be correct to three places after the decimal point.

Unless otherwise specified, the domain of a function f is assumed to be the set of all real numbers x for which $f(x)$ is a real number.

$$W(t)=\begin{cases}\frac{32}{5}+\frac{2}{5}\cos\left(\frac{\pi t}{4}\right) & \text{for } 0\le t\le 4\\ 6+\frac{1}{8}(t-4)^2 & \text{for } 4<t\le 9\end{cases}$$

The depth of water in a zoo aquarium at a certain point is modeled by the function W defined above, where $W(t)$ is measured in feet and time t is measured in hours.

(a) Find $W'(7)$. Using correct units, explain the meaning of $W'(7)$ in the context of the problem.

Please respond on separate paper, following directions from your teacher.

(b) The graph of W is concave up for $2\le t\le 2.5$. Use the line tangent to the graph of W at $t=2$ to show that $W(2.5)\ge 6$.

Please respond on separate paper, following directions from your teacher.

(c) Find $\lim_{t\to 2}\frac{W(t)-t^2-\frac{12}{5}}{t-2}$.

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Please respond on separate paper, following directions from your teacher.

Part A

The first 2 points may be earned with the presence of a correct numerical answer.

The second point may be earned with a maximum of one sign error in $W'(t)$.

The third point requires a time reference, rate, and units. The response is eligible for the third point if the second point is earned AND provided the behavior described is consistent with the sign of the approximation, the numerical value, and the time reference.

Select a point value to view scoring criteria, solutions, and/or examples and to score the response.

0	1	2	3 ✓

The student response accurately includes all three of the criteria below.

- ☐ $W'(t)$
- ☐ $W'(7)$
- ☐ interpretation with unit

Solution:

For $4 < t < 9, W'(t) = \frac{1}{4}(t-4)$. .

$W'(7) = \frac{1}{4}(7-4) = \frac{3}{4}$

At time $t = 7$ hours ,the depth of the water in the zoo aquarium is changing at a rate of $\frac{3}{4}$ foot per hour.

Part B

The second point may be earned with a maximum of one error in $W'(t)$.

For the second and third points, trigonometric function values do not need to be evaluated.

The third point may be earned if the response is consistent with previous results. Select a point value to view scoring criteria, solutions, and/or examples and to score the response.

0	1	2	3	4

The student response accurately includes all four of the criteria below.

- ☐ $W'(t)$
- ☐ $W'(2)$
- ☐ approximation
- ☐ explanation

Solution:

$W(2) = \frac{32}{5} + \frac{2}{5}\cos\left(\frac{2\pi}{4}\right) = \frac{32}{5}$

For $0 < t < 4$,

$W'(t) = -\frac{2}{5}\sin\left(\frac{\pi t}{4}\right) \cdot \frac{\pi}{4} = -\frac{\pi}{10}\sin\left(\frac{\pi t}{4}\right).$

$W'(2) = -\frac{\pi}{10}\sin\left(\frac{2\pi}{4}\right) = -\frac{\pi}{10}$

$W(2.5) \approx \frac{32}{5} - \frac{\pi}{10}(2.5 - 2) = \frac{32}{5} - \frac{\pi}{20}$

Because the graph of W is concave up for $2 \le t \le 2.5$, the tangent line approximation of $W(2.5)$ is an underestimate of the actual value of $W(2.5)$. Therefore, Alt text $W(2.5) > \frac{32}{5} - \frac{\pi}{20} > 6.$

Part C

The first point requires showing that an indeterminate form $\frac{0}{0}$ is present in order to apply L'Hospital's Rule. Any error in mathematical communication (e.g., writing " $\frac{0}{0}$ " or not using limit notation) impacts the first point.

The second point requires substitution of function values. A simplified answer is not required;

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trigonometric function values do not need to be evaluated. Any differentiation or computation error impacts the second point.

Select a point value to view scoring criteria, solutions, and/or examples to score the response.

0	1	2 ✓

The student response accurately includes both of the criteria below.

- ☐ applies L'Hospital's Rule
- ☐ answer

Solution:

$$\lim_{t\to 2}\left(W(t) - t^2 - \frac{12}{5}\right) = W(2) - 2^2 - \frac{12}{5}$$
$$= \frac{32}{5} + \frac{2}{5}\cos\left(\frac{2\pi}{4}\right) - 4 - \frac{12}{5} = 0$$

$$\lim_{t\to 2}(t-2) = 0$$

By L'Hospital's Rule,

$$\lim_{t\to 2}\frac{W(t)-t^2-\frac{12}{5}}{t-2} = \lim_{t\to 2}\frac{W'(t)-2t}{1} = W'(2) - 2\cdot 2$$
$$= -\frac{\pi}{10}\sin\left(\frac{2\pi}{4}\right) - 4 = -\frac{\pi}{10} - 4$$

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2. NO CALCULATOR IS ALLOWED FOR THIS QUESTION.

Show all of your work, even though the question may not explicitly remind you to do so. Clearly label any functions, graphs, tables, or other objects that you use. Justifications require that you give mathematical reasons, and that you verify the needed conditions under which relevant theorems, properties, definitions, or tests are applied. Your work will be scored on the correctness and completeness of your methods as well as your answers. Answers without supporting work will usually not receive credit.

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Unless otherwise specified, answers (numeric or algebraic) need not be simplified. If your answer is given as a decimal approximation, it should be correct to three places after the decimal point.

Unless otherwise specified, the domain of a function f is assumed to be the set of all real numbers x for which $f(x)$ is a real number.

Particle P moves along the x-axis so that its position at time $t > 0$ is given by $x_P(t) = \frac{e^{2-t}-2t}{e^{2-t}+3t}$. A second particle, particle Q, also moves along the x-axis so that its position at time t is given by $x_Q(t) = t^3 - 3t^2 + 5$.

(a) Show that the velocity of particle P at time t is given by $v_P(t) = \frac{-5te^{2-t}-5e^{2-t}}{(e^{2-t}+3t)^2}$.

Please respond on separate paper, following directions from your teacher.

(b) At time $t = 2$, particle Q is at rest. At time $t = 2$, is particle P moving toward particle Q or away from particle Q ? Explain your reasoning.

Please respond on separate paper, following directions from your teacher.

(c) The acceleration of particle Q is given by $a_Q(t)$. Find the value of $a_Q(2)$.

Please respond on separate paper, following directions from your teacher.

(d) Describe the position of particle P and the position of particle Q as t approaches infinity. Show the work that leads to your answers.

Please respond on separate paper, following directions from your teacher.

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Part A

The first point is earned for successful demonstration of the quotient rule.

The second point is earned for a response that arrives at the given expression rather than an algebraically equivalent expression.

Select a point value to view scoring criteria, solutions, and/or examples and to score the response.

0	1	2 ✔

The student response accurately includes both of the criteria below.

- ☐ quotient rule
- ☐ verification

Solution:

$$v_P\ (t) = x'\ (t) = \frac{\left(-e^{2-t}-2\right)\left(e^{2-t}+3t\right)-\left(e^{2-t}-2t\right)\left(-e^{2-t}+3\right)}{\left(e^{2-t}+3t\right)^2}$$

$$= \frac{\left(-e^{4-2t}-3te^{2-t}-2e^{2-t}-6t\right)-\left(-e^{4-2t}+3e^{2-t}+2te^{2-t}-6t\right)}{\left(e^{2-t}+3t\right)^2} = \frac{-5te^{2-t}-5e^{2-t}}{\left(e^{2-t}+3t\right)^2}$$

Part B

The response is eligible for the third point if either of the first 2 points is earned and the answer with reason is consistent with the previous results.

Select a point value to view scoring criteria, solutions, and/or examples and to score the response.

0	1	2	3 ✔

The student response accurately includes all three of the criteria below.

- ☐ $x_P(2)$ and $x_Q(2)$
- ☐ $v_P(2) < 0$
- ☐ answer with explanation

Solution:

$x_P(2) = \frac{1-4}{1+6} = -\frac{3}{7}$;

$x_Q(2) = 1$

$v_P(2) = \frac{-10-5}{7^2} = -\frac{15}{49} < 0$

Because $x_P(2) < x_Q(2)$, particle P is to the left of particle Q at time $t = 2$.

Because $v_P(2) < 0$, P is moving to the left at time $t = 2$.

Because particle Q is at rest at time $t = 2$, , particle P is moving away from particle Q at time $t = 2$.

Part C

Supporting work is not required to earn the point.

Select a point value to view scoring criteria, solutions, and/or examples and to score the response.

0	1 ✔

The student response accurately includes a correct value of $a_Q(2)$

$v_Q(t) = x_Q'(t) = 3t^2 - 6t$
$a_Q(t) = v_Q'(t) = 6t - 6$

$a_Q(2) = 12 - 6 = 6$

Part D

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The first point requires showing that an indeterminate form $\frac{\infty}{\infty}$ is present in order to apply L'Hospital's Rule. Any error in mathematical communication (e.g., writing " $\frac{\infty}{\infty}$ " or not using limit notation) impacts the first point.

For the second and third points, at most 1 out of 2 points may be earned for both correct limits without a connection to the interpretation of position.

Select a point value to view scoring criteria, solutions, and/or examples to score the response.

0	1	2	3 ✓

The student response accurately includes all three of the criteria below.

- ☐ applies L'Hospital's Rule
- ☐ description of position of particle P
- ☐ description of position of particle Q

Solution:

$$\lim_{t\to\infty} e^{2-t} - 2t = -\infty$$

$$\lim_{t\to\infty} e^{2-t} + 3t = \infty$$

By L'Hospital's Rule,

$$\lim_{t\to\infty} x_P(t) = \lim_{t\to\infty} \frac{e^{2-t}-2t}{e^{2-t}+3t} = \lim_{t\to\infty} \frac{-e^{2-t}-2}{-e^{2-t}+3} = -\frac{2}{3}$$

Therefore, particle P approaches $x = -\frac{2}{3}$ as t approaches infinity.

$$\lim_{t\to\infty} x_Q(t) = \lim_{t\to\infty} \left(t^3 - 3t^2 + 5\right) = \infty$$

Therefore, the position of particle Q grows without bound as t approaches infinity.