Unit 3 Free Response Practice

1)

1979 AB2

A function f is defined by $f(x) = xe^{-2x}$ with domain $0 \le x \le 10$.

(a) Find all values of x for which the graph of f is increasing and all values of x for which the graph is decreasing.

(b) Give the x- and y-coordinates of all absolute maximum and minimum points on the graph of f. Justify your answers.

2)

1981 AB3/BC1

Let f be the function defined by $f(x) = 12x^{\frac{2}{3}} - 4x$.

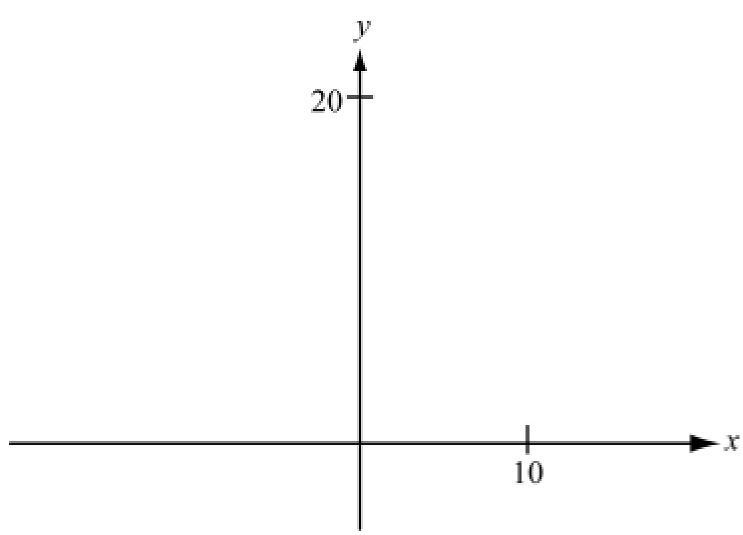
(a) Find the intervals on which f is increasing.

(b) Find the x- and y-coordinates of all relative maximum points.

(c) Find the x- and y-coordinates of all relative minimum points.

(d) Find the intervals on which f is concave downward.

(e) Using the information found in parts (a), (b), (c), and (d), sketch the graph of f on the axes provided.



3)

1981 BC7

Let f be a differentiable function defined for all $x > 0$ such that

(i) $f(1) = 0$,
(ii) $f'(1) = 1$, and
(iii) $\frac{d}{dx}[f(2x)] = f'(x)$, for all $x > 0$.

(a) Find $f'(2)$.

(b) Suppose f' is differentiable. Prove that there is a number c, $2 < c < 4$, such that $f''(c) = -\frac{1}{8}$.

4)

1982 AB5/BC2

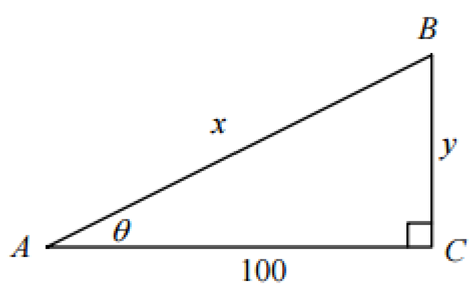
Let f be the function defined by $f(x) = (x^2 + 1)\, e^{-x}$ for $-4 \le x \le 4$.

(a) For what value of x does f reach its absolute maximum? Justify your answer.

(b) Find the x-coordinates of all points of inflection of f. Justify your answer.

5)

1988 BC3



The figure above represents an observer at point A watching balloon B as it rises from point C. The balloon is rising at a constant rate of 3 meters per second and the observer is 100 meters from point C.

(a) Find the rate of change in x at the instant when $y = 50$.

(b) Find the rate of change in the area of right triangle BCA at the instant when $y = 50$.

(c) Find the rate of change in θ at the instant when $y = 50$.

6)

1983 AB1

Let f be the function defined by $f(x) = -2 + \ln(x^2)$.

(a) For what real numbers x is f defined?

(b) Find the zeros of f.

(c) Write an equation for the line tangent to the graph of f at $x = 1$.

7)

1986 AB1

Let f be the function defined by $f(x) = 7 - 15x + 9x^2 - x^3$ for all real numbers x.

(a) Find the zeros of f.

(b) Write an equation of the line tangent to the graph of f at $x = 2$.

(c) Find the x-coordinates of all points of inflection of f. Justify your answer.

8)

1987 BC2

Consider the curve given by the equation $y^3 + 3x^2y + 13 = 0$.

(a) Find $\frac{dy}{dx}$.

(b) Write an equation for the line tangent to the curve at the point $(2, -1)$.

(c) Find the minimum y-coordinate of any point on the curve. Justify your answer.

9)

1985 AB1

Let f be the function defined by $f(x) = \frac{2x-5}{x^2-4}$.

(a) Find the domain of f.

(b) Write an equation for each vertical and each horizontal asymptote for the graph of f.

(c) Find $f'(x)$.

(d) Write an equation for the line tangent to the graph of f at the point $(0, f(0))$.

10)

1984 AB5

The volume V of a cone $\left(V = \frac{1}{3}\pi r^2 h\right)$ is increasing at the rate of 28π cubic units per second. At the instant when the radius r of the cone is 3 units, its volume is 12π cubic units and the radius is increasing at $\frac{1}{2}$ unit per second.

(a) At the instant when the radius of the cone is 3 units, what is the rate of change of the area of its base?

(b) At the instant when the radius of the cone is 3 units, what is the rate of change of its height h?

(c) At the instant when the radius of the cone is 3 units, what is the instantaneous rate of change of the area of its base with respect to its height h?

ANSWER KEY:

1)

1979 AB2
Solution

(a) $f'(x) = e^{-2x} - 2xe^{-2x} = e^{-2x}(1-2x)$

$f'(x) > 0$ when $1-2x > 0$.

The graph of f is increasing for $0 \le x < \frac{1}{2}$.

$f'(x) < 0$ when $1-2x < 0$.

The graph of f is decreasing for $\frac{1}{2} < x \le 10$.

(b) $f'(x) = 0 \Rightarrow e^{-2x}(1-2x) = 0$

There is a critical point when $1-2x = 0$, hence only at $x = \frac{1}{2}$.

$0 < x < \frac{1}{2} \Rightarrow f'(x) > 0$

$\frac{1}{2} < x < 10 \Rightarrow f'(x) < 0$

The graph of f increases and then decreases on the interval $0 \le x \le 10$. Therefore the absolute maximum point is at $\left(\frac{1}{2}, \frac{1}{2e}\right)$.

The absolute minimum value must be at an endpoint.

$f(0) = 0,\ f(10) = \frac{10}{e^{20}}$

Therefore the absolute minimum point is at (0,0)

The absolute maximum can also be justified by using the second derivative test to show that there is a relative maximum at $x = \frac{1}{2}$, then observing that the absolute maximum also occurs at this x value since it is the only critical point in the domain.

2) **1981 AB3/BC1**
Solution

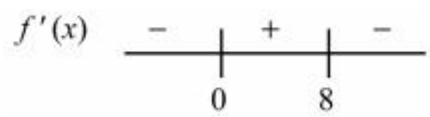
(a) $f(x) = 12x^{2/3} - 4x;\ f'(x) = 8x^{-1/3} - 4$

$(8x^{-1/3} - 4) > 0,\ x > 0 \Rightarrow x < 8$

$(8x^{-1/3} - 4) > 0,\ x < 0 \Rightarrow$ no x satisfies this

or

Critical numbers: $x = 8, x = 0$



Therefore f is increasing on the interval $0 < x < 8$.

(b) 2nd Derivative Test: $f''(x) = -\frac{8}{3}x^{-4/3}$

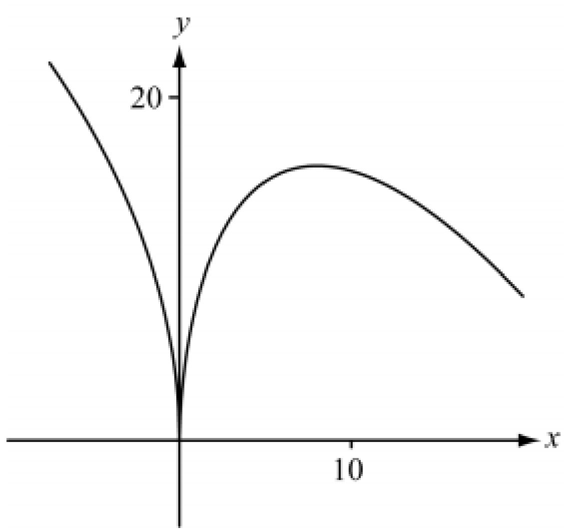
$f''(8) < 0 \Rightarrow$ relative maximum at $(8,16)$

(c) The 2nd Derivative Test cannot be used at $x = 0$ where the second derivative is undefined. Since $f'(x) < 0$ for x just less than 0, and $f'(x) > 0$ for x just greater than 0, there is a relative minimum at $(0,0)$.

(d) $f''(x) = -\frac{8}{3}x^{-4/3} < 0$ if $x \neq 0$

The graph of f is concave down on $(-\infty,0)$ and $(0,+\infty)$.

(e)



3)

1981 BC7
Solution

(a) $\frac{d}{dx}[f(2x)] = f'(x)$ by (iii)

$f'(2x)\cdot 2 = f'(x)$

$f'(2x) = \frac{1}{2}f'(x)$ (*)

$f'(2) = \frac{1}{2}f'(1) = \frac{1}{2}$ by (ii)

(b) There is a c, $2 < c < 4$, so that $f''(c) = \frac{f'(4) - f'(2)}{4-2}$ by the Mean Value theorem.

$f'(2) = \frac{1}{2}$ from (a)

$f'(4) = \frac{1}{4}$ from (*)

Therefore $f''(c) = \frac{\frac{1}{4} - \frac{1}{2}}{4-2} = -\frac{1}{8}$.

4)

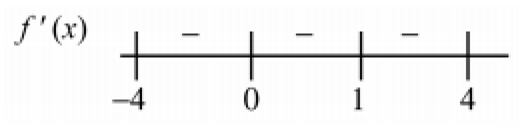
1982 AB5/BC2
Solution

(a) $f(x)=(x^2+1)e^{-x} \qquad -4 \le x \le 4$

$f'(x)=2xe^{-x}-(x^2+1)e^{-x}=-e^{-x}(x+1)^2$

$f'(x) \le 0$ for all x and therefore f is decreasing for all x.

or



Since f is decreasing on the entire interval, the absolute maximum is at $x=-4$.

or

The absolute maximum is at a critical point or an endpoint. There is a critical point at $x=1$.

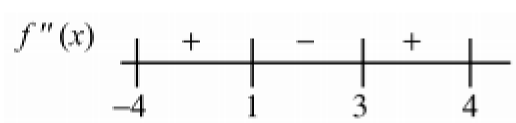
$f(-4)=17e^4$

$f(1)=\dfrac{2}{e}$

$f(4)=\dfrac{17}{e^4}$

Therefore the absolute maximum is at $x=-4$.

(b) $f''(x)=e^{-x}(x-1)^2-e^{-x}\cdot 2(x-1)=e^{-x}(x-1)(x-3)$



$f''(x)>0 \qquad -4<x<1$

$f''(x)<0 \qquad 1<x<3$

$f''(x)>0 \qquad 3<x<4$

The points of inflection are at $x=1$ and $x=3$.

5)

1988 BC3
Solution

(a) $x^2 = y^2 + 100^2$

$2x\frac{dx}{dt} = 2y\frac{dy}{dt}$

At $y = 50$, $x = 50\sqrt{5}$ and $\frac{dx}{dt} = \frac{3 \cdot 50}{50\sqrt{5}} = \frac{3\sqrt{5}}{5}$ m/s

Explicitly: $x = \sqrt{y^2 + 100^2} \Rightarrow \frac{dx}{dt} = \frac{2y}{2\sqrt{y^2+100^2}}\frac{dy}{dt}$

$$= \frac{50}{\sqrt{12500}}(3)$$

$$= \frac{3\sqrt{5}}{5}\text{ m/s}$$

(b) $A = \frac{100y}{2} = 50y$

$\frac{dA}{dt} = 50\frac{dy}{dt} = 50 \cdot 3 = 150$ m/s

(c) $\tan\theta = \frac{y}{100}$

$\sec^2\theta\frac{d\theta}{dt} = \frac{1}{100}\frac{dy}{dt} = \frac{3}{100}$

$\frac{d\theta}{dt} = \frac{3}{100}\cos^2\theta$

At $y = 50$, $\cos\theta = \frac{100}{50\sqrt{5}}$ and therefore $\frac{d\theta}{dt} = \frac{3}{100}\left(\frac{2}{\sqrt{5}}\right)^2 = \frac{3}{125}$ radians/sec.

6)

1983 AB1
Solution

(a) $\ln u$ is defined only for $u > 0$.
$x^2 > 0$ except for $x = 0$.
Therefore $f(x)$ is defined for all $x \neq 0$.

(b) $f(x) = 0$ when $\ln(x^2) = 2$.

$x^2 = e^2$

$|x| = e$

The zeros are $x = \pm e$.

(c) $f'(x) = \frac{2x}{x^2} = \frac{2}{x}$

$f'(1) = \frac{2}{1} = 2$

$f(1) = -2 + \ln(1^2) = -2$

The equation of the tangent line is
$y - (-2) = 2(x-1)$ or $y = 2x - 4$

7)

1986 AB1
Solution

(a) $f(x)=7-15x+9x^2-x^3=-(x-1)^2(x-7)$
The zeros are at $x=1$ and $x=7$.

(b) $f'(x)=-15+18x-3x^2$
$f'(2)=-15+36-12=9$
$f(2)=7-30+36-8=5$
The tangent line is $y-5=9(x-2)$ or $y=9x-13$.

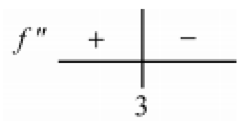
(c) $f''(x)=18-6x$
$18-6x=0,\ x=3$
There is a point of inflection at $x=3$ because

f concave up on $(-\infty,3)$ and concave down on $(3,\infty)$

or

f'' changes sign from positive to negative at $x=3$

or



8)

1987 BC2
Solution

(a) $3y^2y'+3x^2y'+6xy=0$

$$y'=-\frac{6xy}{3x^2+3y^2}=-\frac{2xy}{x^2+y^2}$$

(b) At the point $(2,-1)$, $y'=\dfrac{-(2)(2)(-1)}{4+1}=\dfrac{4}{5}$

The equation of the tangent line is $y+1=\dfrac{4}{5}(x-2)$ or $y=\dfrac{4}{5}x-\dfrac{13}{5}$.

(c) $y'=\dfrac{-2xy}{x^2+y^2}=0\Rightarrow x=0$ or $y=0$

Since y cannot be 0 for any point on the curve, we must have $x=0$. We claim that this gives the minimum y-value on the curve. At $x=0$, $y=-\sqrt[3]{13}$.

$y(y^2+3x^2)=-13\Rightarrow y<0$. Therefore $y'<0$ for $x<0$ and $y'>0$ for $x>0$. Thus $x=0$ does give the minimum value of $y=-\sqrt[3]{13}$.

Non-calculus argument: $y(y^2+3x^2)=-13\Rightarrow y<0$. Therefore $y^3+13=-3x^2y\geq 0$ for all points on the curve. Thus $y\geq-\sqrt[3]{13}$ for all points on the curve. But $y=-\sqrt[3]{13}$ when $x=0$, thus $y=-\sqrt[3]{13}$ is the minimum.

9) **1985 AB1**
Solution

(a) The domain of f is all real numbers except $x=2$ and $x=-2$.

(b) Asymptotes
Vertical: $x=2,\ x=-2$
Horizontal: $y=0$

(c) $f'(x)=\dfrac{2(x^2-4)-2x(2x-5)}{(x^2-4)^2}=\dfrac{-2x^2+10x-8}{(x^2-4)^2}=\dfrac{-2(x-4)(x-1)}{(x^2-4)^2}$

(d) Tangent line at $x=0$

$f(0)=\dfrac{5}{4}$

$f'(0)=-\dfrac{1}{2}$

The equation of the line is

$y-\dfrac{5}{4}=-\dfrac{1}{2}(x-0)$

or

$y=-\dfrac{1}{2}x+\dfrac{5}{4}$

or

$2x+4y=5$

10) **1984 AB5**
Solution

(a) $A=\pi r^2$

When $r=3$, $\dfrac{dA}{dt}=2\pi r\dfrac{dr}{dt}=2\pi\cdot 3\cdot\dfrac{1}{2}=3\pi$

(b) $V=\dfrac{1}{3}\pi r^2 h$

$\dfrac{dV}{dt}=\dfrac{1}{3}\pi r^2\dfrac{dh}{dt}+\dfrac{2}{3}\pi rh\dfrac{dr}{dt}$

$28\pi=\dfrac{1}{3}\pi(9)\dfrac{dh}{dt}+\dfrac{2}{3}\pi(3)(4)\left(\dfrac{1}{2}\right)$

$\dfrac{dh}{dt}=8$

or

$V=\dfrac{1}{3}Ah$

$\dfrac{dV}{dt}=\dfrac{1}{3}A\dfrac{dh}{dt}+\dfrac{1}{3}h\dfrac{dA}{dt}$

$28\pi=\dfrac{1}{3}(9\pi)\dfrac{dh}{dt}+\dfrac{1}{3}4(3\pi)$

$\dfrac{dh}{dt}=8$

(c) $\dfrac{dA}{dh}=\dfrac{\frac{dA}{dt}}{\frac{dh}{dt}}=\dfrac{3\pi}{8}$

or

$A=\pi r^2$

$\dfrac{dA}{dh}=2\pi r\dfrac{dr}{dh}$

$\dfrac{dr}{dh}=\dfrac{\frac{dr}{dt}}{\frac{dh}{dt}}=\dfrac{1/2}{8}=\dfrac{1}{16}$

Therefore $\dfrac{dA}{dh}=2\pi(3)\left(\dfrac{1}{16}\right)=\dfrac{3\pi}{8}$