❶ 2007 #3 (AB but suitable for BC) a,b,c – Calc OK

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
1	6	4	2	5
2	9	2	3	1
3	10	-4	4	2
4	-1	3	6	7

The functions f and g are differentiable for all real numbers, and g is strictly increasing. The table above gives values of the functions and their first derivatives at selected values of x. The function h is given by $h(x) = f(g(x)) - 6$.

(a) Explain why there must be a value r for $1 < r < 3$ such that $h(r) = -5$.

(b) Explain why there must be a value c for $1 < c < 3$ such that $h'(c) = -5$.

❷ 2011B #5 (BC) c – No Calc

t (seconds)	0	10	40	60
$B(t)$ (meters)	100	136	9	49
$v(t)$ (meters per second)	2.0	2.3	2.5	4.6

Ben rides a unicycle back and forth along a straight east-west track. The twice-differentiable function B models Ben's position on the track, measured in meters from the western end of the track, at time t, measured in seconds from the start of the ride. The table above gives values for $B(t)$ and Ben's velocity, $v(t)$, measured in meters per second, at selected times t.

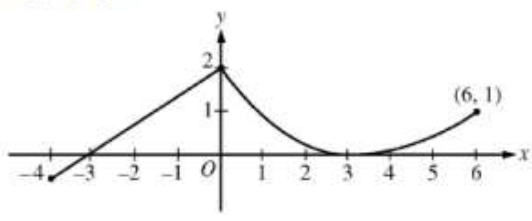
(c) For $40 \le t \le 60$, must there be a time t when Ben's velocity is 2 meters per second? Justify your answer.

❸ 2011B #1 (BC) d – Calc OK

A cylindrical can of radius 10 millimeters is used to measure rainfall in Stormville. The can is initially empty, and rain enters the can during a 60-day period. The height of water in the can is modeled by the function S, where $S(t)$ is measured in millimeters and t is measured in days for $0 \le t \le 60$. The rate at which the height of the water is rising in the can is given by $S'(t) = 2\sin(0.03t) + 1.5$.

(d) During the same 60-day period, rain on Monsoon Mountain accumulates in a can identical to the one in Stormville. The height of the water in the can on Monsoon Mountain is modeled by the function M, where $M(t) = \frac{1}{400}\left(3t^3 - 30t^2 + 330t\right)$. The height $M(t)$ is measured in millimeters, and t is measured in days for $0 \le t \le 60$. Let $D(t) = M'(t) - S'(t)$. Apply the Intermediate Value Theorem to the function D on the interval $0 \le t \le 60$ to justify that there exists a time t, $0 < t < 60$, at which the heights of water in the two cans are changing at the same rate.

❺ 2009B #3 (BC) c – Calc OK

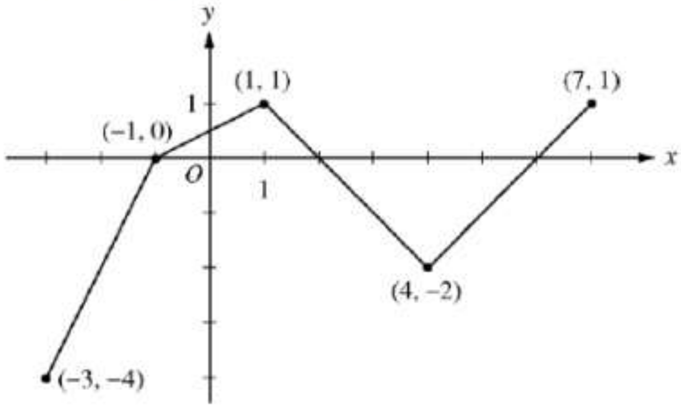


Graph of f

A continuous function f is defined on the closed interval $-4 \le x \le 6$. The graph of f consists of a line segment and a curve that is tangent to the x-axis at $x = 3$, as shown in the figure above. On the interval $0 < x < 6$, the function f is twice differentiable, with $f''(x) > 0$.

(c) Is there a value of a, $-4 \le a < 6$, for which the Mean Value Theorem, applied to the interval $[a, 6]$, guarantees a value c, $a < c < 6$, at which $f'(c) = \frac{1}{3}$? Justify your answer.

❻ 2008B #5 (BC) d – No Calc



Graph of g'

Let g be a continuous function with $g(2) = 5$. The graph of the piecewise-linear function g', the derivative of g, is shown above for $-3 \le x \le 7$.

(d) Find the average rate of change of $g'(x)$ on the interval $-3 \le x \le 7$. Does the Mean Value Theorem applied on the interval $-3 \le x \le 7$ guarantee a value of c, for $-3 < c < 7$, such that $g''(c)$ is equal to this average rate of change? Why or why not?

❶ 2007 #3 (AB but suitable for BC) a,b,c – Calc OK – Scoring Guidelines:

(a) $h(1) = f(g(1)) - 6 = f(2) - 6 = 9 - 6 = 3$
$h(3) = f(g(3)) - 6 = f(4) - 6 = -1 - 6 = -7$
Since $h(3) < -5 < h(1)$ and h is continuous, by the Intermediate Value Theorem, there exists a value r, $1 < r < 3$, such that $h(r) = -5$.

$2: \begin{cases} 1: h(1) \text{ and } h(3) \\ 1: \text{conclusion, using IVT} \end{cases}$

(b) $\frac{h(3) - h(1)}{3 - 1} = \frac{-7 - 3}{3 - 1} = -5$
Since h is continuous and differentiable, by the Mean Value Theorem, there exists a value c, $1 < c < 3$, such that $h'(c) = -5$.

$2: \begin{cases} 1: \frac{h(3) - h(1)}{3 - 1} \\ 1: \text{conclusion, using MVT} \end{cases}$

(c) $w'(3) = f(g(3)) \cdot g'(3) = f(4) \cdot 2 = -2$

$2: \begin{cases} 1: \text{apply chain rule} \\ 1: \text{answer} \end{cases}$

❷ 2011B #5 (BC) c – No Calc – Scoring Guidelines:

(c) Because $\frac{B(60) - B(40)}{60 - 40} = \frac{49 - 9}{20} = 2$, the Mean Value Theorem implies there is a time t, $40 < t < 60$, such that $v(t) = 2$.

$2: \begin{cases} 1: \text{difference quotient} \\ 1: \text{conclusion with justification} \end{cases}$

❸ 2011B #1 (BC) d – Calc OK – Scoring Guidelines:

(d) $D(0) = -0.675 < 0$ and $D(60) = 69.37730 > 0$

Because D is continuous, the Intermediate Value Theorem implies that there is a time t, $0 < t < 60$, at which $D(t) = 0$. At this time, the heights of water in the two cans are changing at the same rate.

$2: \begin{cases} 1: \text{considers } D(0) \text{ and } D(60) \\ 1: \text{justification} \end{cases}$

❺ 2009B #3 (BC) c – Calc OK – Scoring Guidelines:

(c) Yes, $a = 3$. The function f is differentiable on the interval $3 < x < 6$ and continuous on $3 \le x \le 6$.
Also, $\frac{f(6) - f(3)}{6 - 3} = \frac{1 - 0}{6 - 3} = \frac{1}{3}$.
By the Mean Value Theorem, there is a value c, $3 < c < 6$, such that $f'(c) = \frac{1}{3}$.

$2: \begin{cases} 1: \text{answers "yes" and identifies } a = 3 \\ 1: \text{justification} \end{cases}$

❻ 2008B #5 (BC) d – No Calc – Scoring Guidelines:

(d) $\frac{g'(7) - g'(-3)}{7 - (-3)} = \frac{1 - (-4)}{10} = \frac{1}{2}$

No, the MVT does *not* guarantee the existence of a value c with the stated properties because g' is not differentiable for at least one point in $-3 < x < 7$.

$2: \begin{cases} 1: \text{average value of } g'(x) \\ 1: \text{answer "No" with reason} \end{cases}$