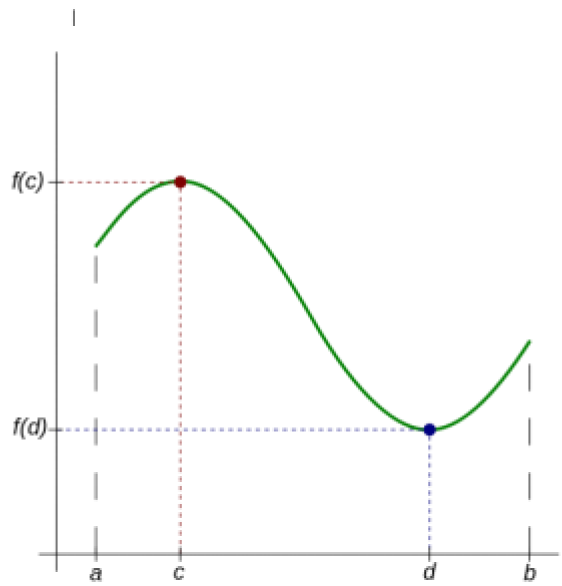
The "Big" Theorems (EVT, IVT, MVT, FTC)

(With special thanks to Lin McMullin)

On the AP Calculus Exams, students should be able to apply the following "Big" theorems though students need not know the proof of these theorems.

The Extreme Value Theorem (EVT)

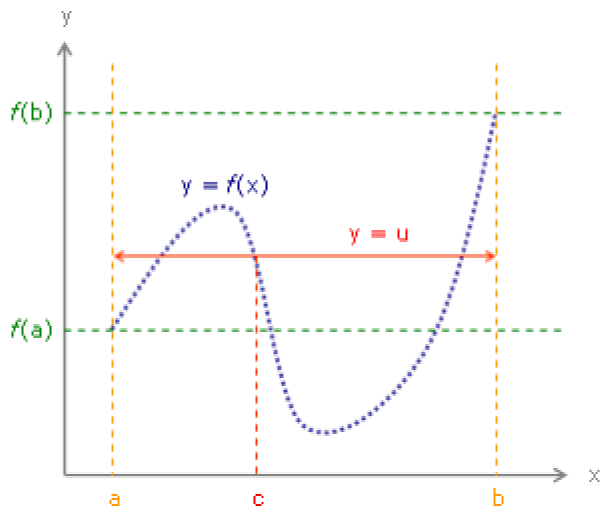
- ***Formal Statement:*** If a function f is continuous on a closed interval $[a, b]$, then:
    1. There exists a number c in $[a, b]$ such that $f(x) \leq f(c)$ for all x in $[a, b]$.
    2. There exists a number d in $[a, b]$ such that $f(x) \geq f(d)$ for all x in $[a, b]$.
- ***Translation:*** If a function f is continuous on a closed interval $[a, b]$, then f takes on a maximum and a minimum value on that interval.
- ***Picture:***



- ***Special Notes:***
    - A function may attain its maximum and minimum value more than once. For example, the maximum value of $y = \sin(x)$ is 1 and it reaches this value many, many times.
    - The extreme values often occur at the endpoint of the domain. That's why it's so important to check the endpoints of an interval when doing a maximization/minimization problem!
    - For a constant function, the maximum and minimum values are equal (in fact, all the values are equal).

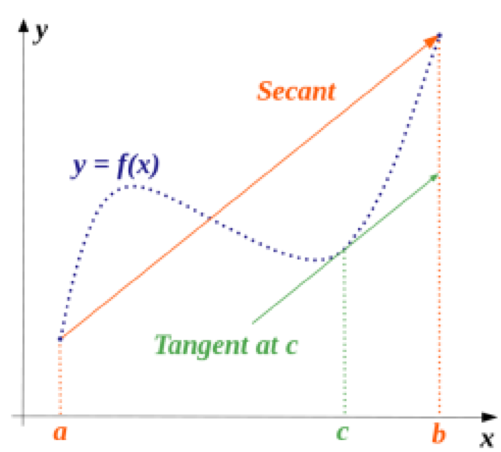
The Intermediate Value Theorem (IVT)

- ***Formal Statement:*** If a function f is continuous on a closed interval $[a, b]$ and $f(a) \neq f(b)$, then for every value of u between $f(a)$ and $f(b)$, there exist at least one value of c in the open interval (a, b) so that $f(c) = u$.
- ***Translation:*** A continuous function takes on all the values between any two of its values.
- ***Picture:***



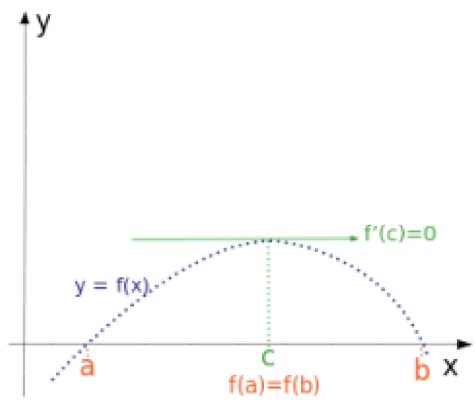
Mean Value Theorem (MVT)

- ***Formal Statement:*** If a function f is continuous on a closed interval $[a, b]$ and differentiable on the open interval (a, b), then there exists a number c in the open interval (a, b) such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.
- ***Translation:*** If a function is continuous and differentiable, somewhere in the interval the tangent line must be parallel to the secant line between the endpoints. In other words, the instantaneous rate of change is equal to the average rate of change.
- ***Picture:***



Rolle's Theorem (a special case of the MVT)

- ***Formal Statement:*** If a function f is continuous on a closed interval $[a, b]$, differentiable on the open interval (a, b), and $f(a) = f(b)$, then there exists a number c in the open interval (a, b) such that $f'(c) = 0$.
- ***Translation:*** If a function is continuous and differentiable, the function must have a place with a horizontal tangent if there are two places where the function takes on the same value. In other words, there must be a relative minimum or maximum between two places where the function takes on the same value.
- ***Picture:***



The Fundamental Theorem of Calculus (FTC)

Assume that $f(x)$, $g(x)$, and $h(x)$ are differentiable functions and that $F(x)$ is an antiderivative of $f(x)$. In other words, $F'(x) = f(x)$.

- ***The First Fundamental Theorem of Calculus (1st FTC)***
  - $\int_a^b f(x)dx = F(b) - F(a)$.
    - **Equivalently:** $\int_a^b f'(x)dx = f(b) - f(a)$ **(sometimes this is called the NET CHANGE)**
    - **Equivalently:** $\int_a^x f'(t)dt = f(x) - f(a)$
  - This yields the incredibly useful formula $f(x) = f(a) + \int_a^x f'(t)dt$. This is used in MANY free response questions!
- ***The Second Fundamental Theorem of Calculus (2nd FTC)***
  - $\frac{d}{dx}\int_a^x f(t)dt = f(x)$
  - **Chain Rule Variation:** $\frac{d}{dx}\int_{g(x)}^{h(x)} f(t)dt = f(h(x)) \cdot h'(x) - f(g(x)) \cdot g'(x)$

Multiple Choice Questions

EVT & IVT

1. **1997 #81 (BC) - Calc OK:** Let f be a continuous function on the closed interval [-3.6]. If $f(-3) = -1$ and $f(6) = 3$, then the Intermediate Value Theorem guarantees that

   a. $f(0) = 0$

   b. $f'(c) = \frac{4}{9}$ for at least one c between -3 and 6.

   c. $-1 \leq f(x) \leq 3$ for all x between -3 and 6.

   d. $f(c) = 1$ for at least one c between -3 and 6.

   e. $f(c) = 0$ for at least one c between -1 and 3.

2. **1998 #91 (AB but suitable for BC) - Calc OK:** Let f be a function that is differentiable on the open interval (1, 10). If $f(2) = -5$, $f(5) = 5$, and $f(9) = -5$, which of the following must be true?

   I. f has at least 2 zeros.
   II. The graph of f has at least one horizontal tangent.
   III. For some c, $2 < c < 5$, $f(c) = 3$.

   a. None
   b. I only
   c. I and II only
   d. I and III only
   e. I, II and III

3. **1998 #26 (AB but suitable for BC) - No Calc:**

x	0	1	2
$f(x)$	1	k	2

   The function f is continuous on the closed interval [0, 2] and has values that are given in the table above. The equation $f(x) = \frac{1}{2}$ must have at least two solutions in the interval [0, 2] if $k =$

   a. 0
   b. $\frac{1}{2}$
   c. 1
   d. 2
   e. 3

MVT

4. **2003 #83 (BC) - Calc OK:**

x	0	1	2	3	4
$f(x)$	2	3	4	3	2

The function f is continuous and differentiable on the closed interval [0,4]. The table above gives selected values of f on this interval. Which of the following statements must be true?

a. The minimum value of f on [0, 4] is 2.

b. The maximum value of f on [0, 4] is 4.

c. $f(x) > 0$ for $0 < x < 4$

d. $f'(x) < 0$ for $2 < x < 4$

e. There exists c, with $0 < c < 4$, for which $f'(c) = 0$.

5. **2003 #80 (AB but suitable for BC) - Calc OK:** The function f is continuous for $-2 \le x \le 1$ and differentiable for $-2 < x < 1$. If $f(-2) = -5$ and $f(1) = 4$, which of the following statements could be false?

a. There exists c, where $-2 < c < 1$, such that $f(c) = 0$.

b. There exists c, where $-2 < c < 1$, such that $f'(c) = 0$.

c. There exists c, where $-2 < c < 1$, such that $f(c) = 3$.

d. There exists c, where $-2 < c < 1$, such that $f'(c) = 3$.

e. There exists c, where $-2 \le c \le 1$ such that $f(c) \ge f(x)$ for all x on the closed interval $-2 \le x \le 1$.

6. **1998 #4 (AB but suitable for BC) - No Calc:** If f is continuous for $a \le x \le b$ and differentiable for $a < x < b$, which of the following could be false?

a. $f'(c) = \dfrac{f(b) - f(a)}{b - a}$ for some c such that $a < c < b$.

b. $f'(c) = 0$ for some c such that $a < c < b$.

c. f has a minimum value on $a \le x \le b$.

d. f has a maximum value on $a \le x \le b$.

e. $\int_a^b f(x)dx$ exists.

7. **2003 #92 (BC) - Calc OK:** Let f be the function defined by $f(x) = x + \ln x$. What is the value of c for which the instantaneous rate of change of f at $x = c$ is the same as the average rate of change of f over $[1,4]$?

a. 0.456
b. 1.244
c. 2.164
d. 2.342
e. 2.452

FTC

8. **2003 #27 (BC) - No Calc:** $\dfrac{d}{dx}\left(\int_0^{x^3} \ln\left(t^2 + 1\right)dt\right) =$

a. $\dfrac{2x^3}{x^6 + 1}$

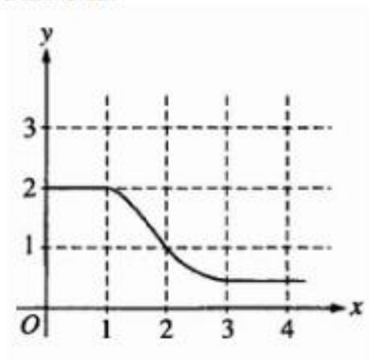
b. $\dfrac{3x^2}{x^6 + 1}$

c. $\ln\left(x^6 + 1\right)$

d. $2x^3 \ln\left(x^6 + 1\right)$

e. $3x^2 \ln\left(x^6 + 1\right)$

9. **1997 #78 (AB but suitable for BC) - Calc OK:**



The graph of f is shown in the figure above. If $\int_1^3 f(x)dx = 2.3$ and $F'(x) = f(x)$, then $F(3) - F(0) =$

a. 0.3

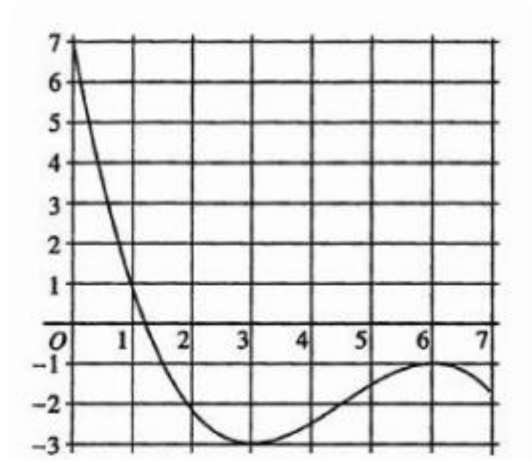
b. 1.3

c. 3.3

d. 4.3

e. 5.3

10. **2003 #18 (BC) - No Calc:**



Graph of f

The graph of the function f shown in the figure above has horizontal tangents at $x = 3$ and $x = 6$. If $g(x) = \int_0^{2x} f(t)dt$, what is the value of $g'(3)$?

a. 0

b. -1

c. -2

d. -3

e. -6

11. **1998 #28 (BC) - No Calc:** $\lim_{x \to 1} \frac{\int_1^x e^{t^2}\,dt}{x^2 - 1}$ is

a. 0
b. 1
c. $\frac{e}{2}$
d. e
e. nonexistent

12. **2003 #80 (BC) - Calc OK:** Insects destroyed a crop at the rate of $\frac{100e^{-0.1t}}{2 - e^{-3t}}$ tons per day, where time t is measured in days. To the nearest ton, how many tons did the insects destroy during the time interval $7 \le t \le 14$?

a. 125
b. 100
c. 88
d. 50
e. 12

13. **1998 #88 (AB but suitable for BC) - Calc OK:** Let $F(x)$ be an antiderivative of $\frac{(\ln x)^3}{x}$. If $F(x) = 0$, then $F(9) =$

a. 0.048
b. 0.144
c. 5.827
d. 23.308
e. 1,640.250

Solutions:

1. D
2. E
3. A
4. E
5. B
6. B
7. C
8. E
9. D
10. C
11. C
12. A
13. C