Student Notes

Prep Session Topic: Area and Volume Problems.

Except for the 2009 BC exam, an area and volume problem has appeared on every one of the AP Calculus exam AB and BC since year 1 (= 1969). They are straightforward and only occasionally have anything out of the ordinary. They are usually the first question on the exam and, as such, are intended to help you get started on the free-response with a familiar yet important application.

You will probably be asked to find the point(s) of intersection of the given functions in order to use them as the limits of integration. Show the equation you are solving, not just the solution. You may use your calculator to find the values. Finding the values is not enough; you do not earn the point until you use the value correctly as a limit of integration somewhere in the problem.

This question usually appears on the graphing calculator allowed section. You are expected to use your calculator to evaluate the definite integrals you set up. Remember to show the integral with limits of integration you are evaluating in standard notation. On the calculator allowed section you do not need to show an antiderivative (and an incorrect antiderivative could cost you a point).

What should you know how to do?

- Solve equations on your calculator. Remember to show the equation you are solving.
- Graph functions on your calculator. Sometimes the graph of the region is not given and you will want to know what it looks like. Be careful in choosing a viewing windov
- Find the area of a region enclosed by two functions over an interval. You will probably have to find the endpoints of the interval to use as limits of integration.
- Find the volume when the region is revolved around an axis or a line using the "washer" or "disk method." If you learned the "method of cylindrical shells", you may use it; however, any volume you are asked for can be done by the washer/disk method.
- Find volumes of solid figures with regular cross sections – squares, equilateral triangles, semi-circles, etc.

Rounding & Simplifying

Answers need not be simplified either numerically or algebraically, so DON'T SIMPLIFY. If you get 1 + 1 leave it! A correct answer simplified incorrectly will lose the point.

The test directions say, "Your final answers must be accurate to three places after the decimal point." This is what "accurate to three places after the decimal point" means: the first three digits after the decimal point must be correct by truncating or rounding. However, you are not required to round or truncate to three-places. Anything written after the third decimal place, right or wrong, is ignored. Thus any of these values for π (which should be left as π) is acceptable: 3.1415926..., 3.141, 3.142, 3.1416, 3.141789..., 3.1425643.... Once again: DON'T ROUND, a correct answer that is then rounded incorrectly loses the answer point.

Student Notes

DON'T ROUND. Carry at least 4 or 5 decimal places into ANY computation. If a value is rounded to three places and that number is used in the computation resulting in a final answer is not correct to three places after the decimal point, you will lose the "answer point."

Multiple-choice questions:

Underlined questions are calculator required; others are no calculator allowed

Area: 1, 2, <u>3</u>, <u>4</u>, <u>5</u>
Volume of revolution: 6,
Volume by slicing: <u>7</u>, 8, <u>9</u>, <u>10</u>, <u>11</u>

WATCH and LISTEN to the multiple-choice questions being solved
→ change # to see others

Go to http://tinyurl.com/NMSI-Math-4 Click on the "Full Screen" arrow. Then click anywhere on the page to see and hear from that point on.

Click anywhere to go back anytime.

Free-response questions included

2008 AB 1

2000 AB 1

1998 AB 1

Area - Volume Problems

Multiple Choice
Identify the choice that best completes the statement or answers the question.

Underlined questions are calculator required; others are no calculator.

Area: 1, 2, <u>3</u>, <u>4</u>, <u>5</u>
Volume of revolution: 6,
Volume by slicing: <u>7</u>, 8, <u>9</u>, <u>10</u>, <u>11</u>

1. The area of the region enclosed by the graph of $y = x^2 + 1$ and the line $y = 5$ is

a. $\frac{14}{3}$ b. $\frac{16}{3}$ c. $\frac{28}{3}$ d. $\frac{32}{3}$ e. 8π

2. What is the area of the region between the graphs of $y = x^2$ and $y = -x$ from $x = 0$ to $x = 2$?

a. $\frac{2}{3}$ b. $\frac{8}{3}$ c. 4 d. $\frac{14}{3}$ e. $\frac{16}{3}$

3. Let R be the region enclosed by the graph of $y = 1 + \ln\left(\cos^4 x\right)$ the x-axis, and the lines $x = -\frac{2}{3}$ and $x = \frac{2}{3}$. The closest integer approximation of the area of R is

a. 0 b. 1 c. 2 d. 3 e. 4

4. What is the area of the region in the first quadrant enclosed by the graphs of $y = \cos x$, $y = x$, and the y-axis?

a. 0.127 b. 0.385 c. 0.400 d. 0.600 e. 0.947

5. If $0 \le k < \frac{\pi}{2}$ and the area under the curve $y = \cos x$ from $x = k$ to $x = \frac{\pi}{2}$ is 0.1, then $k =$

a. 1.471 b. 1.414 c. 1.277 d. 1.120 e. 0.436

6. If the region enclosed by the y-axis, the line $y = 2$, and the curve $y = \sqrt{x}$ is revolved about the y-axis, the volume of the solid generated is

a. $\frac{32\pi}{5}$ b. $\frac{16\pi}{3}$ c. $\frac{16\pi}{5}$ d. $\frac{8\pi}{3}$ e. π

Calculator

7. The base of a solid S is the region enclosed by the graph of $y = \sqrt{\ln x}$, the line $x = e$, and the x-axis. If the cross sections of S perpendicular to the x-axis are squares, then the volume of S is

a. $\frac{1}{2}$ b. $\frac{2}{3}$ c. 1 d. 2 e. $\frac{1}{3}(e^3 - 1)$

8. The base of a solid is the region in the first quadrant enclosed by the graph of $y = 2 - x^2$ and the coordinate axes. If every cross section of the solid perpendicular to the y-axis is a square, the volume of the solid is given by

a. $\pi\int_0^2 (2-y)^2 dy$

b. $\int_0^2 (2-y) dy$

c. $\pi\int_0^{\sqrt{2}} (2-x^2)^2 dx$

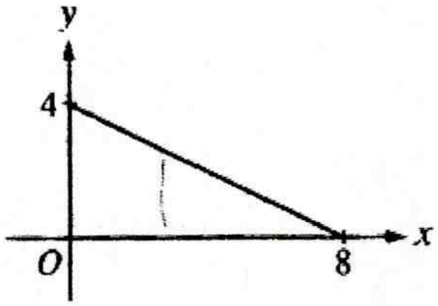
d. $\int_0^{\sqrt{2}} (2-x^2)^2 dx$

e. $\int_0^{\sqrt{2}} (2-x^2) dx$

9. The region bounded by the graph of $y = 2x - x^2$ and the x-axis is the base of a solid. For this solid, each cross section perpendicular to the x-axis is an equilateral triangle. What is the volume of the solid?

a. 1.333
b. 1.067
c. 0.577
d. 0.462
e. 0.267

10.



The base of a solid is a region in the first quadrant bounded by the x-axis, the y-axis, and the line $x + 2y = 8$, as shown in the figure above. If cross sections of the solid perpendicular to the x-axis are semicircles, what is the volume of the solid?

a. 12.566

b. 14.661

c. 16.755

d. 67.021

e. 134.041

11. The base of a solid is the region in the first quadrant bounded by the y-axis, the graph of $y = \tan^{-1} x$, the horizontal line $y = 3$, and the vertical line $x = 1$. For this solid, each cross section perpendicular to the x-axis is a square. What is the volume of the solid?

a. 2.561

b. 6.612

c. 8.046

d. 8.755

e. 20.773

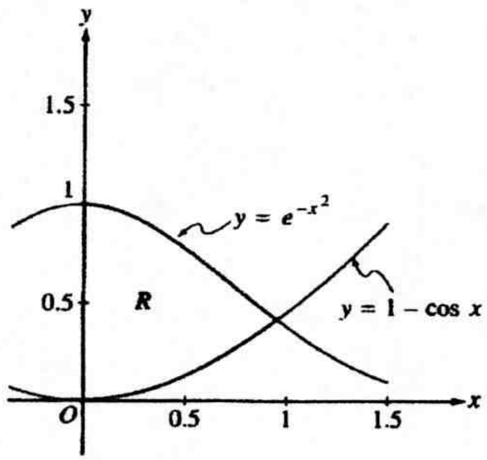
CALCULUS AB

SECTION II, Part A

Time—45 minutes

Number of problems—3

A graphing calculator is required for some problems or parts of problems.



1. Let R be the shaded region in the first quadrant enclosed by the graphs of $y = e^{-x^2}$, $y = 1 - \cos x$, and the y-axis, as shown in the figure above.
   (a) Find the area of the region R.
   (b) Find the volume of the solid generated when the region R is revolved about the x-axis.
   (c) The region R is the base of a solid. For this solid, each cross section perpendicular to the x-axis is a square. Find the volume of this solid.

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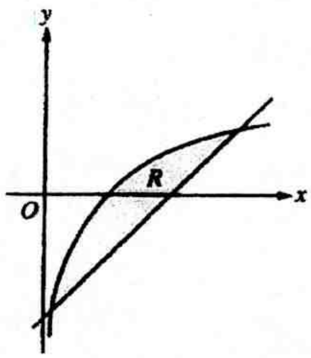
CALCULUS AB
SECTION II, Part A
Time—45 minutes
Number of problems—3

A graphing calculator is required for some problems or parts of problems.



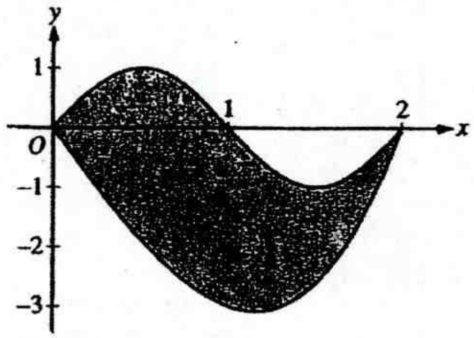
1. Let R be the shaded region bounded by the graph of $y = \ln x$ and the line $y = x - 2$, as shown above.

   (a) Find the area of R.

   (b) Find the volume of the solid generated when R is rotated about the horizontal line $y = -3$.

   (c) Write, but do not evaluate, an integral expression that can be used to find the volume of the solid generated when R is rotated about the y-axis.

WRITE ALL WORK IN THE PINK EXAM BOOKLET.



Let R be the region bounded by the graphs of $y = \sin(\pi x)$ and $y = x^3 - 4x$, as shown in the figure above.

(a) Find the area of R.

(b) The horizontal line $y = -2$ splits the region R into two parts. Write, but do not evaluate, an integral expression for the area of the part of R that is below this horizontal line.

(c) The region R is the base of a solid. For this solid, each cross section perpendicular to the x-axis is a square. Find the volume of this solid.

(d) The region R models the surface of a small pond. At all points in R at a distance x from the y-axis, the depth of the water is given by $h(x) = 3 - x$. Find the volume of water in the pond.

1998 AP Calculus AB Scoring Guidelines

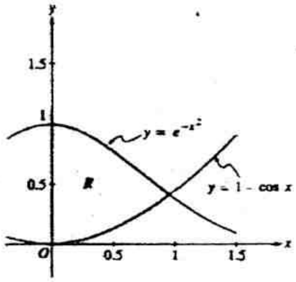
1. Let R be the region bounded by the x- axis, the graph of $y = \sqrt{x}$, and the line $x = 4$.

   (a) Find the area of the region R.

   (b) Find the value of h such that the vertical line $x = h$ divides the region R into two regions of equal area.

   (c) Find the volume of the solid generated when R is revolved about the x–axis.

   (d) The vertical line $x = k$ divides the region R into two regions such that when these two regions are revolved about the x–axis, they generate solids with equal volumes. Find the value of k.



Let R be the shaded region in the first quadrant enclosed by the graphs of $y = e^{-x^2}$, $y = 1 - \cos x$, and the y-axis, as shown in the figure above.

(a) Find the area of the region R.

(b) Find the volume of the solid generated when the region R is revolved about the x-axis.

(c) The region R is the base of a solid. For this solid, each cross section perpendicular to the x-axis is a square. Find the volume of this solid.

Region R

$e^{-x^2} = 1 - \cos x$ at $x = 0.941944 = A$

1 : Correct limits in an integral in (a), (b), or (c).

(a) Area $= \int_0^A (e^{-x^2} - (1 - \cos x))\,dx$

$= 0.590$ or 0.591

2 $\begin{cases} 1: \text{ integrand} \\ 1: \text{ answer} \end{cases}$

(b) Volume $= \pi \int_0^A \left(\left(e^{-x^2}\right)^2 - (1 - \cos x)^2 \right) dx$

$= 0.55596\pi = 1.746$ or 1.747

3 $\begin{cases} 2: \text{ integrand and constant} \\ \quad <-1> \text{ each error} \\ 1: \text{ answer} \end{cases}$

(c) Volume $= \int_0^A \left(e^{-x^2} - (1 - \cos x)\right)^2 dx$

$= 0.461$

3 $\begin{cases} 2: \text{ integrand} \\ \quad <-1> \text{ each error} \\ \quad \text{Note: 0/2 if not of the form} \\ \qquad k\int_c^d (f(x) - g(x))^2\,dx \\ 1: \text{ answer} \end{cases}$

MC ① D ② D ③ B ④ C ⑤ D ⑥ A ⑦ C ⑧ B ⑨ D ⑩ C ⑪ B

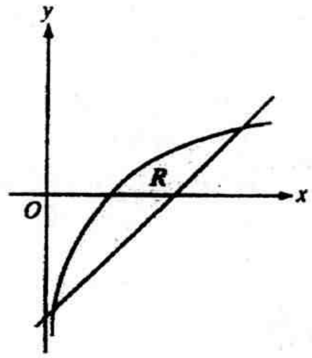
Question 1

Let R be the shaded region bounded by the graph of $y = \ln x$ and the line $y = x - 2$, as shown above.

(a) Find the area of R.

(b) Find the volume of the solid generated when R is rotated about the horizontal line $y = -3$.

(c) Write, but do not evaluate, an integral expression that can be used to find the volume of the solid generated when R is rotated about the y-axis.



$\ln(x) = x - 2$ when $x = 0.15859$ and 3.14619.
Let $S = 0.15859$ and $T = 3.14619$

(a) Area of $R = \int_S^T (\ln(x) - (x-2))\,dx = 1.949$

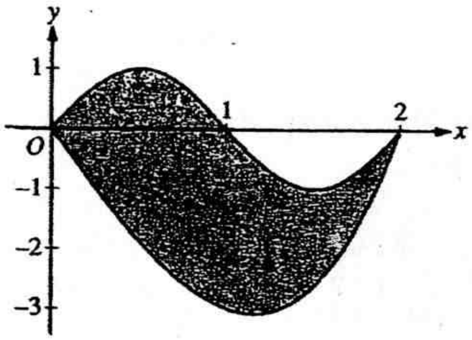
$3: \begin{cases} 1: \text{integrand} \\ 1: \text{limits} \\ 1: \text{answer} \end{cases}$

(b) Volume $= \pi \int_S^T \left((\ln(x)+3)^2 - (x-2+3)^2\right) dx$
$= 34.198$ or 34.199

$3: \begin{cases} 2: \text{integrand} \\ 1: \text{limits, constant, and answer} \end{cases}$

(c) Volume $= \pi \int_{S-2}^{T-2} \left((y+2)^2 - \left(e^y\right)^2\right) dy$

$3: \begin{cases} 2: \text{integrand} \\ 1: \text{limits and constant} \end{cases}$



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(d) The region R models the surface of a small pond. At all points in R at a distance x from the y-axis, the depth of the water is given by $h(x) = 3 - x$. Find the volume of water in the pond.

(a) $\sin(\pi x) = x^3 - 4x$ at $x = 0$ and $x = 2$ Area $= \int_0^2 \left(\sin(\pi x) - \left(x^3 - 4x\right)\right) dx = 4$	$3: \begin{cases} 1: \text{limits} \\ 1: \text{integrand} \\ 1: \text{answer} \end{cases}$
(b) $x^3 - 4x = -2$ at $r = 0.5391889$ and $s = 1.6751309$ The area of the stated region is $\int_r^s \left(-2 - \left(x^3 - 4x\right)\right) dx$	$2: \begin{cases} 1: \text{limits} \\ 1: \text{integrand} \end{cases}$
(c) Volume $= \int_0^2 \left(\sin(\pi x) - \left(x^3 - 4x\right)\right)^2 dx = 9.978$	$2: \begin{cases} 1: \text{integrand} \\ 1: \text{answer} \end{cases}$
(d) Volume $= \int_0^2 (3 - x)\left(\sin(\pi x) - \left(x^3 - 4x\right)\right) dx = 8.369$ or 8.370	$2: \begin{cases} 1: \text{integrand} \\ 1: \text{answer} \end{cases}$

1998 AP Calculus AB Scoring Guidelines

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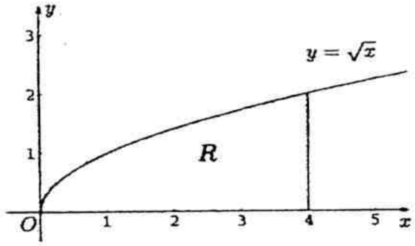
   (d) The vertical line $x = k$ divides the region R into two regions such that when these two regions are revolved about the x-axis, they generate solids with equal volumes. Find the value of k.

(a)



$$A = \int_0^4 \sqrt{x}\,dx = \frac{2}{3}x^{3/2}\Big|_0^4 = \frac{16}{3} \text{ or } 5.333$$

2 { 1: $A = \int_0^4 \sqrt{x}\,dx$; 1: answer }

(b) $\int_0^h \sqrt{x}\,dx = \frac{8}{3}$ –or– $\int_0^h \sqrt{x}\,dx = \int_h^4 \sqrt{x}\,dx$

$\frac{2}{3}h^{3/2} = \frac{8}{3}$ $\frac{2}{3}h^{3/2} = \frac{16}{3} - \frac{2}{3}h^{3/2}$

$h = \sqrt[3]{16}$ or 2.520 or 2.519

2 { 1: equation in h ; 1: answer }

(c) $V = \pi \int_0^4 (\sqrt{x})^2\,dx = \pi \frac{x^2}{2}\Big|_0^4 = 8\pi$

or 25.133 or 25.132

3 { 1: limits and constant ; 1: integrand ; 1: answer }

(d) $\pi \int_0^k (\sqrt{x})^2\,dx = 4\pi$ –or– $\pi \int_0^k (\sqrt{x})^2\,dx = \pi \int_k^4 (\sqrt{x})^2\,dx$

$\pi \frac{k^2}{2} = 4\pi$ $\pi \frac{k^2}{2} = 8\pi - \pi \frac{k^2}{2}$

$k = \sqrt{8}$ or 2.828

2 { 1: equation in k ; 1: answer }