p-Series: (Packet Key)

1. converges if $p > 1$
2. diverges if $0 < p < 1$
3. diverges if $p = 1$ (harmonic series)

A. $\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}}$

$\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ $p = \frac{3}{2} > 1$

$\therefore$ converges

B. $\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n}}$

$\sum_{n=1}^{\infty} \frac{1}{n^{1/3}}$ $p = \frac{1}{3} < 1$

$\therefore$ diverges

C. $\sum_{n=1}^{\infty} \frac{1}{\sqrt[4]{n}}$ $p = \frac{1}{4} < 1$

$\therefore$ diverges

D. Not a p-series!
(Either geo or integral)

Geo: $\sum_{n=1}^{\infty} \left(\frac{1}{e}\right)^n$ $r = \frac{1}{e} < 1$

$\therefore$ converges

Integral:

$$\int_1^{\infty} e^{-x}\,dx = \lim_{b \to \infty} \int_1^{b} e^{-x}\,dx$$

$$= \lim_{b \to \infty} \left(-e^{-x} \Big|_1^b\right)$$

$$= \lim_{b \to \infty} \left(-e^{-b} - (-e^{-1})\right)$$

$$= \frac{1}{e}$$

$\therefore$ converges

E. Either Telescopic or Integral:

$$\sum_{n=2}^{\infty} \frac{1}{n^2-1}$$

$$\frac{1}{x^2-1} = \frac{1}{(x+1)(x-1)} = \frac{A}{x+1} + \frac{B}{x-1}$$

$$1 = A(x-1) + B(x+1)$$

$$x = 1: 1 = 2B \quad B = \frac{1}{2}$$

$$x = -1: 1 = -2A \quad A = -\frac{1}{2}$$

$$\frac{1}{x^2-1} = \frac{-\frac{1}{2}}{x+1} + \frac{\frac{1}{2}}{x-1} = \frac{1}{2}\left(\frac{1}{x-1} - \frac{1}{x+1}\right)$$

Telescopic:

$$\sum_{n=2}^{\infty} \frac{1}{2}\left(\frac{1}{x-1} - \frac{1}{x+1}\right)$$

$$= \frac{1}{2}\left[\left(1-\cancel{\frac{1}{3}}\right)+\left(\frac{1}{2}-\cancel{\frac{1}{4}}\right) + \left(\cancel{\frac{1}{3}}-\cancel{\frac{1}{5}}\right)+\left(\cancel{\frac{1}{4}}-\frac{1}{6}\right) + \left(\cancel{\frac{1}{5}}-\frac{1}{7}\right)+\ldots\right]$$

$$\frac{1}{2}\left(1+\frac{1}{2}\right)$$

$$\frac{1}{2}\left(\frac{3}{2}\right)$$

$\therefore$ converges to $\frac{3}{4}$

Integral: f is continuous, positive, and decreasing

$$\frac{1}{2}\int_2^{\infty} \frac{1}{x-1}dx - \frac{1}{2}\int_2^{\infty} \frac{1}{x+1}dx$$

$$\lim_{b\to\infty} \frac{1}{2}\int_2^{b} \frac{1}{x-1}dx - \frac{1}{2}\int_2^{b} \frac{1}{x+1}dx$$

$$\lim_{b\to\infty} \frac{1}{2}\ln|x-1| - \frac{1}{2}\ln|x+1| \Big|_2^b$$

$$\lim_{b\to\infty} \frac{1}{2}\ln\left|\frac{x-1}{x+1}\right| \Big|_2^b$$

$$\lim_{b\to\infty} \frac{1}{2}\ln\left|\frac{b-1}{b+1}\right| - \frac{1}{2}\ln\left|\frac{1}{3}\right|$$

$$= -\frac{1}{2}\ln\frac{1}{3} = \frac{\ln 3}{2}$$

$\therefore$ converges

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F. $\sum_{n=3}^{\infty} \frac{1}{[\ln(\ln n)][n(\ln n)]}$

Integral Test: f is continuous, positive, and decreasing

$\int_3^{\infty} \frac{1}{\ln(\ln x) \cdot x \ln x} dx$

$u = \ln(\ln x)$

$du = \frac{1}{\ln x} \cdot \frac{1}{x} dx = \frac{1}{x \ln x} dx$

$\lim_{b \to \infty} \int_3^{b} \frac{1}{\ln(\ln x) \cdot x \ln x} dx$

$\int \frac{1}{u} du = \ln|u| =$
$\ln|\ln(\ln x)|$

$\lim_{b \to \infty} \ln|\ln(\ln x)| \Big|_3^b$

$\lim_{b \to \infty} \ln|\ln(\ln b)| - \ln|\ln(\ln 3)|$

$\therefore$ series diverges