1978 AB5/BC1

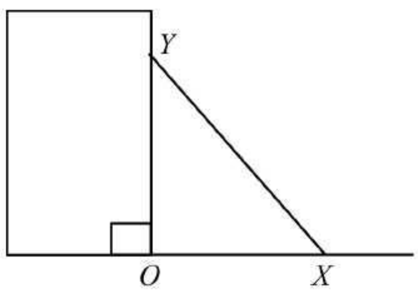
Given the curve $x^2 - xy + y^2 = 9$.

(a) Write a general expression for the slope of the curve.

(b) Find the coordinates of the points on the curve where the tangents are vertical.

(c) At the point $(0,3)$ find the rate of change in the slope of the curve with respect to x.

1982 AB4



A ladder 15 feet long is leaning against a building so that end X is on level ground and end Y is on the wall as shown in the figure. X is moved away from the building at the constant rate of $\frac{1}{2}$ foot per second.

(a) Find the rate in feet per second at which the length OY is changing when X is 9 feet from the building.

(b) Find the rate of change in square feet per second of the area of triangle XOY when X is 9 feet from the building.

1984 AB5

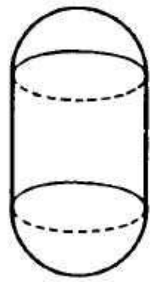
The volume V of a cone $\left(V = \frac{1}{3}\pi r^2 h\right)$ is increasing at the rate of 28π cubic units per second. At the instant when the radius r of the cone is 3 units, its volume is 12π cubic units and the radius is increasing at $\frac{1}{2}$ unit per second.

(a) At the instant when the radius of the cone is 3 units, what is the rate of change of the area of its base?

(b) At the instant when the radius of the cone is 3 units, what is the rate of change of its height h?

(c) At the instant when the radius of the cone is 3 units, what is the instantaneous rate of change of the area of its base with respect to its height h?

1985 AB5/BC2



The balloon shown is in the shape of a cylinder with hemispherical ends of the same radius as that of the cylinder. The balloon is being inflated at the rate of 261π cubic centimeters per minute. At the instant the radius of the cylinder is 3 centimeters., the volume of the balloon is 144π cubic centimeters and the radius of the cylinder is increasing at the rate of 2 centimeters per minute. (The volume of a cylinder is $\pi r^2 h$ and the volume of a sphere is $\frac{4}{3}\pi r^3$).

(a) At this instant, what is the height of the cylinder?

(b) At this instant, how fast is the height of the cylinder increasing?

1992 AB4/BC1

Consider the curve defined by the equation $y + \cos y = x + 1$ for $0 \le y \le 2\pi$.

(a) Find $\frac{dy}{dx}$ in terms of y.

(b) Write an equation for each vertical tangent to the curve.

(c) Find $\frac{d^2y}{dx^2}$ in terms of y.

1978 AB5/BC1
Solution

(a) Implicit differentiation gives

$$2x - xy' - y + 2yy' = 0$$

$$(2y - x)y' = y - 2x$$

$$y' = \frac{y - 2x}{2y - x}$$

(b) There is a vertical tangent when $2y - x = 0$, so $x = 2y$. Substituting into the equation of the curve gives $(2y)^2 - (2y)y + y^2 = 9$, or $3y^2 = 9$. Therefore $y = \pm\sqrt{3}$ and the two points on the curve where the tangents are vertical are $(2\sqrt{3}, \sqrt{3})$ and $(-2\sqrt{3}, -\sqrt{3})$.

(c) $y'' = \dfrac{(2y - x)(y' - 2) - (y - 2x)(2y' - 1)}{(2y - x)^2}$

At the point $(0, 3)$, $y' = \dfrac{3 - 0}{6 - 0} = \dfrac{1}{2}$ and so $y'' = \dfrac{(6-0)\left(\frac{1}{2} - 2\right) - (3 - 0)(1 - 1)}{(6 - 0)^2} = -\dfrac{1}{4}$

Alternatively, one can use implicit differentiation a second time to get

$$2 - xy'' - y' - y' + 2yy' + 2(y')^2 = 0$$

Substituting $x = 0$, $y = 3$, and $y' = \dfrac{1}{2}$ gives

$$2 - 0 - \frac{1}{2} - \frac{1}{2} + 6y'' + 2\left(\frac{1}{4}\right) = 0 \Rightarrow 6y'' = -\frac{3}{2} \Rightarrow y'' = -\frac{1}{4}$$

1982 AB4
Solution

(a) $x^2 + y^2 = 15^2$

Implicit: $2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$

$9 \cdot \frac{1}{2} + 12\frac{dy}{dt} = 0$

$\frac{dy}{dt} = -\frac{3}{8}$

(b) $A = \frac{1}{2}xy$

Implicit: $\frac{dA}{dt} = \frac{1}{2}\left(x\frac{dy}{dt} + y\frac{dx}{dt}\right)$

$\frac{dA}{dt} = \frac{1}{2}\left(9 \cdot \left(-\frac{3}{8}\right) + 12 \cdot \frac{1}{2}\right)$

$\frac{dA}{dt} = \frac{21}{16}$

1984 AB5
Solution

(a) $A = \pi r^2$

When $r = 3$, $\frac{dA}{dt} = 2\pi r \frac{dr}{dt} = 2\pi \cdot 3 \cdot \frac{1}{2} = 3\pi$

(b) $V = \frac{1}{3}\pi r^2 h$

$\frac{dV}{dt} = \frac{1}{3}\pi r^2 \frac{dh}{dt} + \frac{2}{3}\pi r h \frac{dr}{dt}$

$28\pi = \frac{1}{3}\pi(9)\frac{dh}{dt} + \frac{2}{3}\pi(3)(4)\left(\frac{1}{2}\right)$

$\frac{dh}{dt} = 8$

or

$V = \frac{1}{3}Ah$

$\frac{dV}{dt} = \frac{1}{3}A\frac{dh}{dt} + \frac{1}{3}h\frac{dA}{dt}$

$28\pi = \frac{1}{3}(9\pi)\frac{dh}{dt} + \frac{1}{3}4(3\pi)$

$\frac{dh}{dt} = 8$

(c) $\frac{dA}{dh} = \frac{\frac{dA}{dt}}{\frac{dh}{dt}} = \frac{3\pi}{8}$

or

$A = \pi r^2$

$\frac{dA}{dh} = 2\pi r \frac{dr}{dh}$

$\frac{dr}{dh} = \frac{\frac{dr}{dt}}{\frac{dh}{dt}} = \frac{1/2}{8} = \frac{1}{16}$

Therefore $\frac{dA}{dh} = 2\pi(3)\left(\frac{1}{16}\right) = \frac{3\pi}{8}$

1985 AB5/BC2
Solution

(a) $V = \pi r^2 h + \frac{4}{3}\pi r^3$

$144\pi = \pi(3)^2 h + \frac{4}{3}\pi(3)^3$

$h = 12$

At this instant, the height is 12 centimeters.

(b) $\frac{dV}{dt} = \pi r^2 \frac{dh}{dt} + 2\pi rh\frac{dr}{dt} + 4\pi r^2 \frac{dr}{dt}$

$261\pi = \pi(3)^2 \frac{dh}{dt} + 2\pi(3)(12)(2) + 4\pi(3)^2(2)$

$\frac{dh}{dt} = 5$

At this instant, the height is increasing at the rate of 5 centimeters per minute.

1992 AB4/BC1
Solution

(a) $\frac{dy}{dx} - \sin y \frac{dy}{dx} = 1$

$\frac{dy}{dx}(1 - \sin y) = 1$

$\frac{dy}{dx} = \frac{1}{1 - \sin y}$

(b) $\frac{dy}{dx}$ undefined when $\sin y = 1$

$y = \frac{\pi}{2}$

$\frac{\pi}{2} + 0 = x + 1$

$x = \frac{\pi}{2} - 1$

(c) $\frac{d^2 y}{dx^2} = \frac{d\left(\frac{1}{1 - \sin y}\right)}{dx}$

$= \frac{-\left(-\cos y \frac{dy}{dx}\right)}{(1 - \sin y)^2}$

$= \frac{\cos y\left(\frac{1}{1 - \sin y}\right)}{(1 - \sin y)^2}$

$= \frac{\cos y}{(1 - \sin y)^3}$