Taylor Polynomial Worksheet

1. NO CALCULATORS: Let f be the function given by $f(t) = \dfrac{4}{1+t^2}$ and G be the function given by $G(x) = \int_0^x f(t)\,dt$.

   a. Find the first four non-zero terms and the general term for the power series expansion of $f(t)$ about $t = 0$.

   b. Find the first four non-zero terms and the general term for the power series expansion of $G(x)$ about $x = 0$.

   c. Find the interval of convergence of the power series in part (b). (Your solution must include an analysis that justifies your answer.)

NO CALCULATORS:

a. Find the first five terms in the Taylor series about $x = 0$ for $f(x) = \dfrac{1}{1-2x}$.

b. Find the interval of convergence for the series in part (a).

c. Use partial fractions and the result from part a to find the first five terms in the Taylor series about $x = 0$ for $g(x) = \dfrac{1}{(1-2x)(1-x)}$.

a. Find the first four nonzero terms in the Taylor series expansion about $x = 0$ for $f(x) = \sqrt{1+x}$.

b. Use the results found in part (a) to find the first four nonzero terms in the Taylor series expansion about $x = 0$ for $g(x) = \sqrt{1+x^3}$.

c. Find the first four nonzero terms in the Taylor series expansion about $x = 0$ for the function h such that $h'(x) = \sqrt{1+x^3}$ and $h(0) = 4$.

Taylor Series Wksht

1. $f(t) = \frac{4}{1+t^2}$ $\qquad h(x) = \frac{1}{1-x} = 1 - x + x^2 - x^3 + \dots$

$G(x) = \int_0^x f(t)\,dt$

a) $f(t) = 4(1 - t^2 + t^4 - t^6 + \dots + (-1)^n t^{2n} + \dots)$

$= 4 - 4t^2 + 4t^4 - 4t^6 + \dots + (-1)^n \cdot 4t^{2n} + \dots$

b) $G(x) = \int_0^x \frac{4}{1+t^2}\,dt$

$= 4 \tan^{-1} t \Big|_0^x$

$= 4 \tan^{-1} x$ $\Big\}$ optional

ok

$= 4\left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots + \frac{(-1)^n x^{2n-1}}{2n-1} + \dots\right)$

$= 4x - \frac{4x^3}{3} + \frac{4x^5}{5} - \frac{4x^7}{7} + \dots + \frac{(-1)^n 4 x^{2n-1}}{2n-1} + \dots$

c) $\sum_{k=1}^{\infty} \frac{4(-1)^k x^{2k-1}}{2k-1}$

$\lim_{k\to\infty} \left| \frac{4(-1)^{k+1} x^{2k+1}}{2k+1} \cdot \frac{2k-1}{4(-1)^k x^{2k-1}} \right|$

$= \lim_{k\to\infty} \left| \frac{2k-1}{2k+1} \cdot x^2 \right|$ $\qquad |x^2| < 1$

$\qquad -1 < x < 1$

$= |x^2| \lim_{k\to\infty} \left| \frac{2k-1}{2k+1} \right| \to 1$

EP: $x = -1 \Rightarrow \sum_{k=1}^{\infty} \frac{4(-1)^k \cancel{(-1)^{2k-1}}}{2k-1} = \sum_{k=1}^{\infty} \frac{4(-1)^{k+1}}{2k-1}$ converges by AST.

$x = 1 \Rightarrow \sum_{k=1}^{\infty} \frac{4(-1)^k \cancel{(1)^{2k-1}}}{2k-1} = \sum_{k=1}^{\infty} \frac{4(-1)^k}{2k-1}$ converges by AST

Interval of convergence: $\boxed{-1 \le x \le 1}$

5) $f(x) = (1-2x)^{-1}$ $\qquad f(0) = 1$

$f'(x) = -(1-2x)^{-2}(-2)$ $\qquad f'(0) = 2$

$= 2(1-2x)^{-2}$

$f''(x) = -4(1-2x)^{-3}(-2)$ $\qquad f''(0) = 8$

$= 8(1-2x)^{-3}$

$f'''(x) = -24(1-2x)^{-4}(-2)$ $\qquad f^{(3)}(0) = 48$

$= 48(1-2x)^{-4}$

$f^{(4)}(x) = 384(1-2x)^{-5}$ $\qquad f^{(4)}(0) = 384$

$f(x) = 1 + 2(x) + \frac{8}{2!}x^2 + \frac{48}{3!}x^3 + \frac{384}{4!}x^4 + \dots$

a) $f(x) = 1 + 2x + 4x^2 + 8x^3 + 16x^4 + \dots$

b) $\lim_{n\to\infty} \left| \frac{2^{n+1} x^{n+1}}{2^n x^n} \right| = |x| \lim_{n\to\infty} 2 = 2|x| < 1$

$|x| < \frac{1}{2}$

at $x = -\frac{1}{2}$: $\sum_{n=0}^{\infty} 2^n \left(-\frac{1}{2}\right)^n = \sum_{n=0}^{\infty} (-1)$ diverges

at $x = \frac{1}{2}$ $\quad \sum_{n=0}^{\infty} 2^n \left(\frac{1}{2}\right)^n = \sum_{n=0}^{\infty} 1$ diverges

interval of converge: $\boxed{-\frac{1}{2} < x < \frac{1}{2}}$

5 c) $g(x) = \frac{1}{(1-2x)(1-x)} = \frac{A}{1-2x} + \frac{B}{1-x}$

$A(1-x) + B(1-2x) = 1$

$A+B + x(-A-2B) = 1$

$A+B=1 \qquad -A-2B=0$

$+) \quad A+B=1$

$-B = 1$

$B=-1 \qquad A=2$

$g(x) = \frac{2}{1-2x} - \frac{1}{1-x}$

$= 2(1+2x+4x^2+8x^3+16x^4) - (1+x+x^2+x^3+x^4)$

$= 1+3x+7x^2+15x^3+31x^4$

6)
$f(x) = (1+x)^{\frac{1}{2}} \qquad f(0)=1$

$f'(x) = \frac{1}{2}(1+x)^{-\frac{1}{2}} \qquad f'(0) = \frac{1}{2}$

$f''(x) = -\frac{1}{4}(1+x)^{-\frac{3}{2}} \qquad f''(0) = -\frac{1}{4}$

$f'''(x) = \frac{3}{8}(1+x)^{-\frac{5}{2}} \qquad f'''(0) = \frac{3}{8}$

a) $f(x) = 1 + \frac{1}{2}(x) - \frac{1}{4}\cdot\frac{1}{2!}x^2 + \frac{3}{8}\cdot\frac{1}{3!}x^3 + \dots$

$= 1 + \frac{x}{2} - \frac{1}{8}x^2 + \frac{1}{16}x^3 + \dots$

b) $g(x) = (1+x^3)^{\frac{1}{2}} = f(x^3)$

$= 1 + \frac{x^3}{2} - \frac{1}{8}x^6 + \frac{1}{16}x^9 + \dots$

c) $h'(x) = (1+x^3)^{\frac{1}{2}} = g(x)$

$h(x) = \int g(x)\,dx + C$

$= x + \frac{1}{2}\cdot\frac{1}{4}x^4 - \frac{1}{56}x^7 + \frac{1}{160}x^{10} + \dots + C$

$h(0) = 4 = C$

$\boxed{h(x) = 4 + x + \frac{1}{8}x^4 - \frac{1}{56}x^7 + \dots}$