Problem Set #3 Name: ______________________

Series and Sequences

1. Which of the following series is absolutely convergent?

(A) $\sum_{k=0}^{\infty}(-1)^k \frac{k+3}{k+\sqrt{k}}$

(B) $\sum_{k=0}^{\infty}(-1)^k \frac{3}{\sqrt{k}}$

(C) $\sum_{k=0}^{\infty}(-1)^{k+1} \frac{\sqrt{k}}{k+3}$

(D) $\sum_{k=0}^{\infty}(-1)^k \frac{3}{k+3}$

(E) $\sum_{k=0}^{\infty}(-1)^{k+1} \frac{3}{k\sqrt{k}}$

2. The power series $\sum_{k=1}^{\infty}\frac{x^k}{k}$ converges for which values of x?

(A) $x = 0$
(B) $-1 < x < 1$
(C) $-1 \le x < 1$
(D) $-1 \le x \le 1$
(E) x is any real number.

3. Which of the following series is the power series expansion for $f(x) = x(\cos x - 1)$?

(A) $x - \frac{x^3}{2} + \frac{x^5}{24} - \cdots$

(B) $-x^3 + x^5 - x^7 + \cdots$

(C) $\frac{x^3}{2} - \frac{x^5}{24} + \frac{x^7}{720} - \cdots$

(D) $-\frac{x^3}{2} + \frac{x^5}{24} - \frac{x^7}{720} + \cdots$

(E) $1 - \frac{x^2}{2} + \frac{x^4}{24} - \cdots$

4. What are all values of a for which the series $\sum_{k=0}^{\infty}\left(\frac{5}{9-a}\right)^k$ converges?

(A) $a < 4$
(B) $4 < a < 14$
(C) $a < 9$
(D) $a < 9$ or $a > 9$
(E) $a < 4$ or $a > 14$

5. The Maclaurin series for $f(x) = \frac{1}{1+x^2}$ is $\sum_{k=0}^{\infty}(-1)^k x^{2k}$. What is the Maclaurin series for $g(x) = \tan^{-1} x$?

(A) $\sum_{k=0}^{\infty}(-1)^k (2k) x^{2k-1}$

(B) $\sum_{k=0}^{\infty}(-1)^k \frac{x^{2k+1}}{2k+1}$

(C) $\sum_{k=0}^{\infty}(-1)^{k+1} \frac{x^{2k+1}}{2k}$

(D) $\sum_{k=0}^{\infty}(-1)^k (2k+1) x^{2k+1}$

(E) $\sum_{k=0}^{\infty}(-1)^{k-1} \frac{x^{2k-1}}{2k}$

6. The Maclaurin series $\sum_{k=0}^{\infty}(-9)^k \frac{x^{4k}}{(2k)!}$ represents which function below?

(A) $\cos(3x^2)$
(B) $\sin(3x^2)$
(C) $\cos(9x^4)$
(D) $\tan^{-1}(3x)$
(E) e^{-3x^2}

7. If the first five terms of the Taylor expansion for $f(x)$ about $x = 0$ are $3 - 7x + \frac{5}{2}x^2 + \frac{3}{4}x^3 - 6x^4$, then $f'''(0) =$

(A) $\frac{1}{8}$

(B) $\frac{3}{4}$

(C) $\frac{9}{2}$

(D) 6
(E) 8

8. Which of the following series diverge?

I. $\sum_{k=0}^{\infty}\frac{k^{\frac{3}{2}}+1}{5k^2+7}$

II. $\sum_{k=2}^{\infty}(-1)^k \frac{1}{\ln(k)}$

III. $\sum_{k=0}^{\infty}(-1)^k \left(\frac{4}{3}\right)^k$

(A) I only
(B) II only
(C) I and II only
(D) I and III only
(E) I, II, and III

You may use a calculator on the following questions

9. The sixth degree term of the Taylor series expansion for $f(x) = e^{-\frac{1}{2}x^2}$ about $x = 0$ has coefficient

(A) $-\frac{1}{48}$

(B) $-\frac{1}{6}$

(C) $\frac{1}{720}$

(D) $\frac{1}{6}$

(E) $-\frac{1}{4608}$

10. Let represent the Taylor Polynomial of a degree n about x = 0 for . If $T_n(2)$ is used to approximate the value of $f(2) = e^{-2}$, then which of the following expressions is NOT less than —?

(A) $f(2) - T_6(2)$
(B) $T_6(2) - f(2)$
(C) $f(2) - T_5(2)$
(D) $T_5(2) - f(2)$
(E) $f(2) - T_7(2)$

FREE-RESPONSE QUESTION

You may use a calculator for this question.

Let $f(x)$ be a function that is differentiable for all x. The Taylor expansion for $f(x)$ about $x = 0$ is given by $T(x) = \sum_{k=0}^{\infty} (-1)^k \frac{(k+1)x^{3k}}{k!}$. The first four nonzero terms of $T(x)$ are given by $T_4(x) = 1 - 2x^3 + \frac{3x^6}{2!} - \frac{4x^9}{3!}$.

a. Show that $T(x)$ converges for all x.

b. Let $W(x)$ be the Taylor expansion for $x^2 f'(x)$ about $x = 0$. Find the general term for $W(x)$ and find $W_3(x)$.

c. $f(0.5) \approx T_4(0.5)$ and $f(1) \approx T_4(1)$. Find the values of $T_4(0.5)$ and $T_4(1)$.

d. Which value is smaller, $|f(0.5) - T_4(0.5)|$ or $|f(1) - T_4(1)|$? Give a reason for your answer.

Answer Sheet

1. E	2. C	3. D	4. E	5. B
6. A	7. C	8. D	9. A	10.C

a. Using the Ratio Test for absolute convergence,

$$\lim_{k\to\infty}\left\lvert\frac{\frac{(k+2)x^{3(k+1)}}{(k+1)!}}{\frac{(k+1)x^{3k}}{k!}}\right\rvert<1\Rightarrow$$

$$\lim_{k\to\infty}\left\lvert\frac{(k+2)x^{3k+3}}{(k+1)!}\cdot\frac{k!}{(k+1)x^{3k}}\right\rvert<1\Rightarrow$$

$$\lim_{k\to\infty}\left\lvert\frac{k+2}{k+1}\cdot\frac{x^3}{k+1}\right\rvert<1\Rightarrow 1\cdot 0<1$$ which is true for all x.

Therefore $T(x)$ converges for all real numbers.

2: { 1: correct use of Ratio Test for Absolute Convergence; 1: answer }

b. $f'(x)=-3\cdot 2x^2+\frac{6\cdot 3x^5}{2!}-\frac{9\cdot 4x^8}{3!}+\cdot\cdot\cdot=$

$\sum_{k=1}^{\infty}(-1)^k\frac{3k(k+1)x^{3k-1}}{k!}$. Therefore

$W(x)=x^2f'(x)=$

$\sum_{k=1}^{\infty}(-1)^k\frac{3k(k+1)x^{3k+1}}{k!}$ and $W_3(x)=-6x^4+9x^7-6x^{10}$.

4: { 1: expanded $f'(x)$; 1: general term of $f'(x)$; 1: general term of $W(x)$; 1: $W_3(x)$ }

c. Using direct substitution or a graphing calculator evaluation, $T_4(0.5)=0.772$ and $T_4(1)=-0.167$.

1: both answers correct

d)Because T(x) is an alternating series, each difference above is smaller than the fifth term of the expansion. Therefore $|f(0.5) - T_4(0.5)|$ is smaller. Alternatively, $|f(0.5) - T_4(0.5)|$ is smaller because 0.5 is closer than 1 to the center x =0 of the interval of convergence. Therefore the series converges to f(x) more quickly.

1 point answer
1 point reason