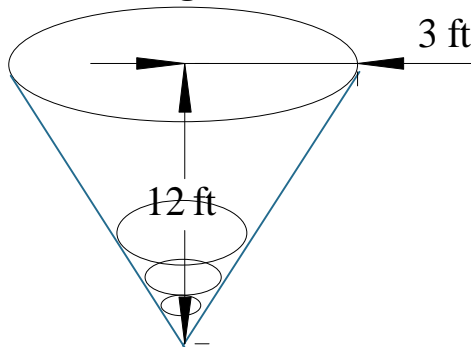


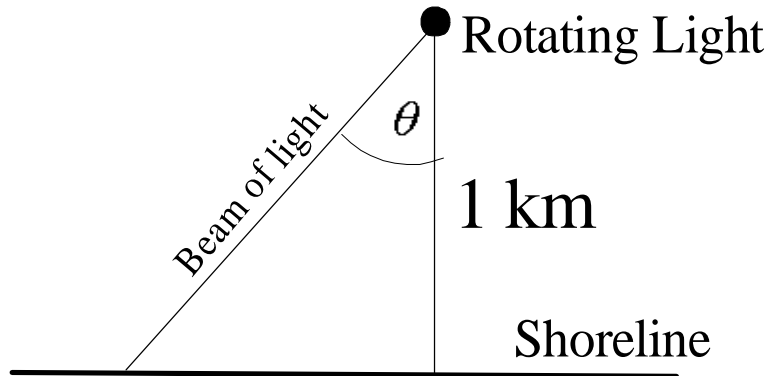
## Related Rates Problem Set

(Revised Oct. 20, 2010)

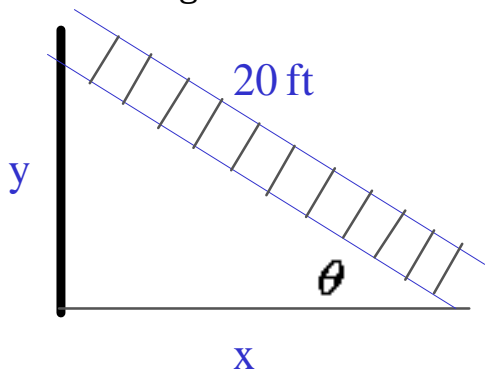
1. If  $z^2 = x^2 + y^2$ ,  $dx/dt = 2$ , and  $dy/dt = 3$ , find  $dz/dt$  when  $x = 5$  and  $y = 12$ . Assume that  $z \geq 0$ .
2. A particle moves along the curve  $y = \sqrt{1 + x^3}$ . As it reaches the point  $(2, 3)$ , the  $y$ -coordinate is increasing at a rate of 4 cm/sec. How fast is the  $x$ -coordinate of the point changing at this instant?
3. Suppose  $PV = k$  where  $k$  is a constant and  $\frac{dP}{dt} = 2$  when  $P = 4$  and  $V = 5$ . Find  $\frac{dV}{dt}$ .
4. Two cars start moving from the same point. One travels south at 60 m/hr and the other travels west at 25 m/hr. At what rate is the distance between the cars increasing two hours later?
5. A plane flying horizontally at an altitude of 1 mile and a speed of 500 mph passes over a radar station. Find the rate at which the distance from the plane to the station is increasing when it is 2 miles away from the station.
6. A conical water tower has a height of 12 ft and a radius of 3 ft. Water is pumped into the tank at a rate of  $4 \text{ ft}^3/\text{min}$ . How fast is the water level rising when the water level is 6 ft?  
See the diagram.



7. A light in a lighthouse 1 km offshore from a straight shoreline is rotating at 2 revolutions per minute ( $d\theta/dt = 2 \text{ rev}/\text{min}$ ). See the drawing below. How fast is the beam moving along the shoreline when it passes the point 1/2 km from the point opposite the lighthouse?



8. At noon, ship A is 100 km west of ship B. Ship A is sailing south at 35 km/hr and ship B is sailing north at 25 km/hr. How fast is the distance between the ships changing at 4:00 P.M.?
9. An 20 ft long ladder is leaning against a wall. The bottom of the ladder is sliding away from the wall at a rate of 2.5 ft/sec. See the diagram.



- How fast is the top of the ladder sliding down the wall when  $x = 12$  ft. Note that this rate is  $|dy/dt|$ .
- How fast is the angle  $\theta$  changing when  $x = 12$  ft?
- How fast is the area of the triangle changing when  $x = 12$  ft?

## Solutions

1. If  $z^2 = x^2 + y^2$  then

$$2z \frac{dz}{dt} = 2x \frac{dx}{dt} + 2y \frac{dy}{dt} \Rightarrow$$
$$z' = \frac{x dx/dt + y dy/dt}{z} = \frac{(5)(2) + (12)(3)}{\sqrt{5^2 + 12^2}} = \frac{46}{13}.$$

2. If  $y = \sqrt{1 + x^3}$  then  $\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} \Rightarrow$

$$\frac{dy}{dt} = \frac{1}{2}(1 + x^3)^{-1/2}(3x^2) \frac{dx}{dt} \Rightarrow$$

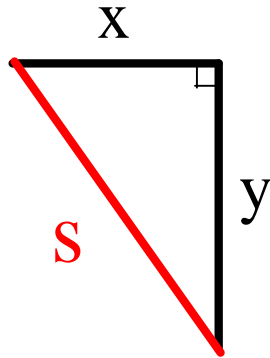
$$4 = \frac{1}{2(1 + 2^3)^{1/2}}(3 \cdot 2^2) \frac{dx}{dt} \Rightarrow$$

$$4 = \frac{12}{6} \frac{dx}{dt} \Rightarrow \frac{dx}{dt} = 2 \text{ cm/sec}.$$

3. If  $PV = k$  then  $\frac{dPV}{dt} = \frac{dk}{dt} \Rightarrow$

$$\frac{dP}{dt}V + P \frac{dV}{dt} = 0 \Rightarrow 2(5) + 4 \frac{dV}{dt} = 0 \Rightarrow$$
$$\frac{dV}{dt} = -\frac{10}{4} = -\frac{5}{2}.$$

4. Look at the situation  $t$  hrs later, where  $t$  is an arbitrary time. The 1st car has moved south (say a distance  $y$ ) and the 2nd car has moved west (say a distance  $x$ ). We have the triangle below.



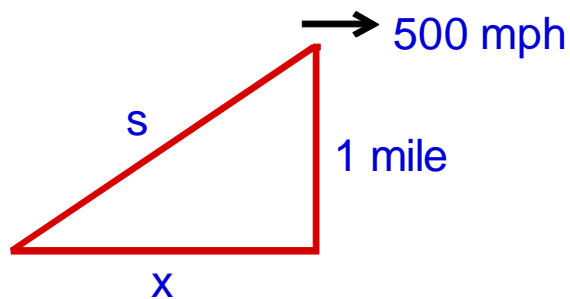
$$x^2 + y^2 = s^2 \leftarrow \text{differentiate both sides with respect to } t$$

$$2xx' + 2yy' = 2ss' \Rightarrow s' = \frac{xx' + yy'}{s}$$

where  $x' = 25$  and  $y' = 60$  and  $s'$  is the unknown in the problem. Note that 2 hours later,  $x = 50$  m,  $y = 120$  m, and  $s = \sqrt{50^2 + 120^2} = 130$  m. Therefore

$$s' = \frac{(50)(25) + (120)(60)}{130} = 65 \text{ m/hr.}$$

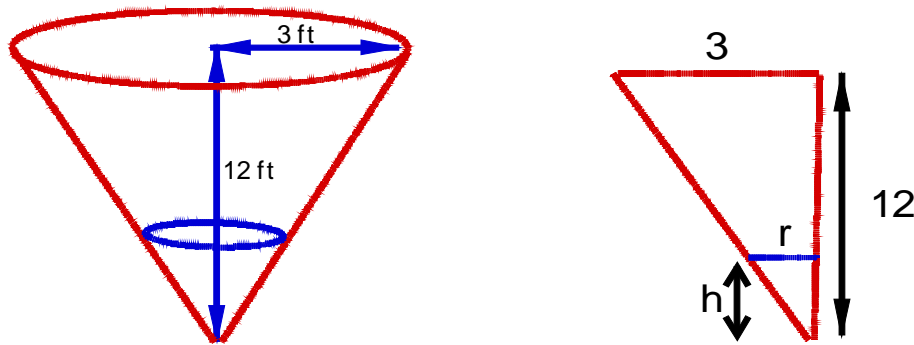
5.



$$s^2 = x^2 + 1^2 \Rightarrow 2ss' = 2xx' \Rightarrow s' = \frac{xx'}{s} = \frac{(500)(\sqrt{4-1})}{2}$$

$$250\sqrt{3} = 433 \text{ mph.}$$

6. Let  $h$  be the height of the water level and let  $r$  be the radius of the cone of water.



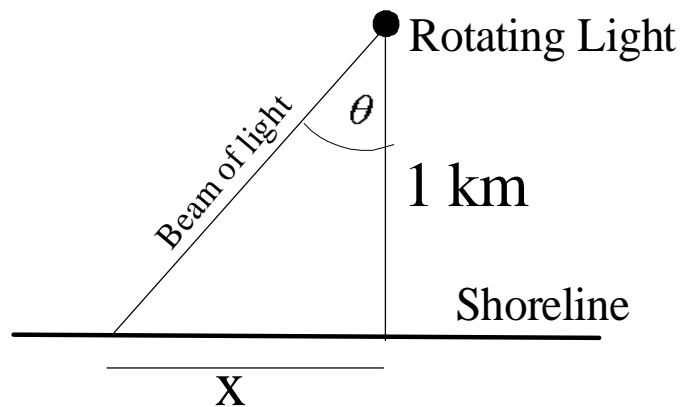
Because of similar triangles,  $\frac{12}{3} = \frac{h}{r} \Rightarrow h = 4r \Rightarrow r = \frac{h}{4}$

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi \left(\frac{h}{4}\right)^2 h = \frac{\pi}{48}h^3$$

$$\frac{dV}{dt} = \frac{dV}{dh} \cdot \frac{dh}{dt} = \frac{1}{16}\pi h^2 \frac{dh}{dt} = 4 \Rightarrow \frac{dh}{dt} = \frac{64}{\pi h^2} = \frac{64}{\pi(6)^2} =$$

$$\frac{64}{36\pi} = \frac{16}{9\pi} \text{ ft/min .}$$

7. Let  $x$  be the length shown below. We want  $\frac{dx}{dt}$  when  $x = \frac{1}{2}$ .

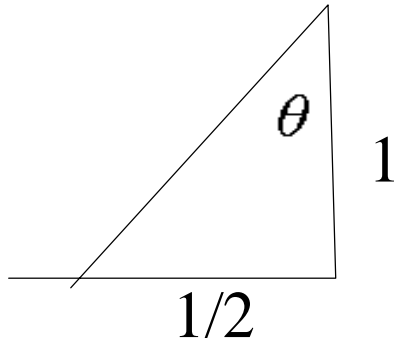


$$\text{We know } \frac{d\theta}{dt} = \frac{2 \text{ revolutions}}{\text{min}} = \frac{2(2\pi \text{ radians})}{\text{min}} = \frac{4\pi \text{ radians}}{\text{min}} .$$

From the drawing we see that  $\tan \theta = \frac{x}{1} = x$ , or better  $x = \tan \theta$ . Differentiate with respect to time.

$$\frac{dx}{dt} = \sec^2 \theta \frac{d\theta}{dt} = \sec^2 \theta \cdot 4\pi.$$

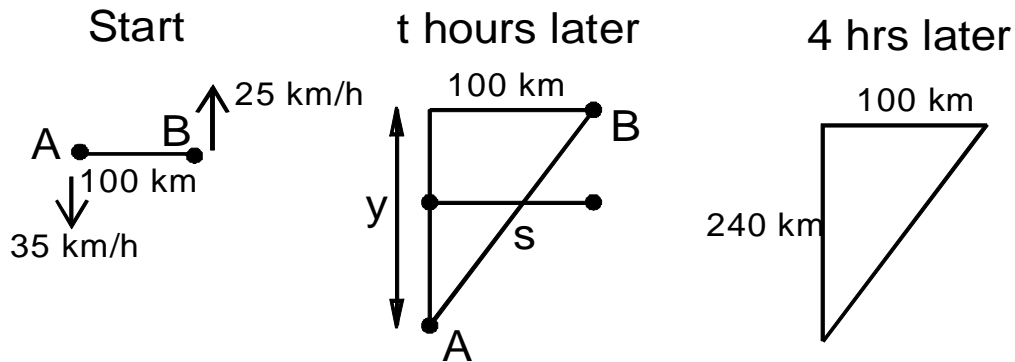
$$\frac{1}{4} + 1 = h^2 \Rightarrow h = \sqrt{\frac{5}{4}}.$$



$$\sec^2 \theta = \frac{5/4}{1} = 5/4$$

$$\frac{dx}{dt} = \frac{5}{4} \cdot 4\pi = 5\pi = 15.71 \text{ km/min}.$$

8.



$$s^2 = y^2 + 100^2$$

$$2ss' = 2yy' \Rightarrow s' = \frac{yy'}{s}.$$

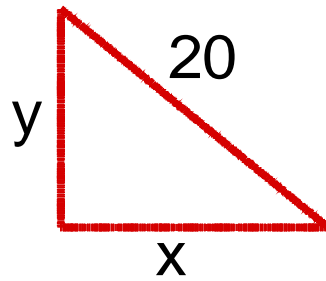
$$4 \text{ hrs later } y = 4(25) + 4(35) = 100 + 140 = 240 \text{ km}$$

$$y' = 25 + 35 = 60 \text{ km/hr}$$

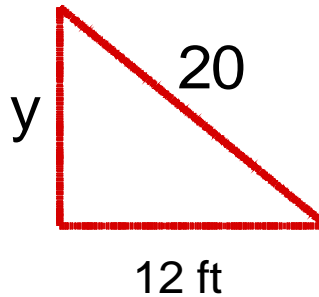
$$s' = \frac{(240)(60)}{\sqrt{240^2 + 100^2}} = 55.385 \text{ km/hr}$$

9.

Triangle at arbitrary time



Triangle when  $x=12$

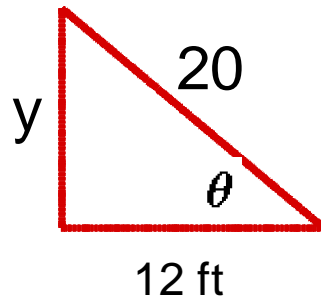


$$\begin{aligned} \text{a) } x^2 + y^2 = 400 &\Rightarrow 2xx' + 2yy' = 0 \Rightarrow \\ y' &= \frac{-xx'}{y} . \end{aligned}$$

$$\begin{aligned} \text{When } x = 12, y^2 + 144 = 400 &\Rightarrow y = \sqrt{400 - 144} = 16 . \\ y' &= \frac{-(12)(2.5)}{16} = -1.875 \text{ ft/sec} \end{aligned}$$

Therefore the top of the ladder is sliding down at a speed of 1.875 ft/sec.

b)



$$\begin{aligned} \sin \theta = \frac{y}{20} &\Rightarrow y = 20 \sin \theta \Rightarrow \frac{dy}{dt} = 20 \cos \theta \frac{d\theta}{dt} \\ \Rightarrow \frac{d\theta}{dt} &= \frac{y'}{20 \cos \theta} = \frac{-1.875}{20(12/20)} = -0.156 \text{ radians/sec} \end{aligned}$$

$$\begin{aligned} \text{c) } A = \frac{1}{2}xy &\Rightarrow \frac{dA}{dt} = \frac{1}{2} \frac{dx}{dt}y + \frac{1}{2}x \frac{dy}{dt} = \\ &\frac{1}{2}(2.5)(16) + \frac{1}{2}(12)(-1.875) = 8.75 \text{ ft}^2/\text{sec} . \end{aligned}$$