Day 3 – More related rates

Objective: Use basic math formulas and derivatives to solve problems

1. A water tank has the shape of an inverted circular cone with base radius 2 m and height 4m. If water is being pumped into the tank at a rate of 2 m³/min, find the rate at which the water level is rising when the water is 3 m deep.

$\frac{dv}{dt} = 2\,m^3/min$

$V = \frac{1}{3}\pi r^2 h$

$V = \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h$

$V = \frac{1}{3}\pi\left(\frac{h^2}{4}\right)h = \frac{1}{12}\pi h^3$

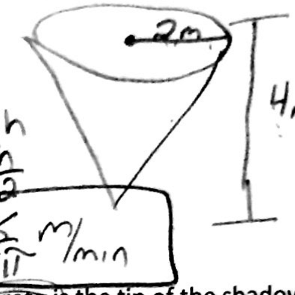
$\frac{dv}{dt} = \frac{1}{4}\pi h^2 \frac{dh}{dt}$

$2 = \frac{1}{4}\pi(3)^2 \frac{dh}{dt}$

$2r = h$

$r = \frac{h}{2}$

$\boxed{\frac{dh}{dt} = \frac{8}{9\pi}\ m/min}$



2. A man 6ft tall man is walking at the rate of 3 ft/s toward a streetlight 18 ft high. At what rate is the tip of the shadow moving? At what rate is the shadow length changing?

$\frac{6}{18} = \frac{x}{x+y}$

$18x = 6x + 6y$

$\frac{12x}{6} = \frac{6y}{6}$

$2x = y$

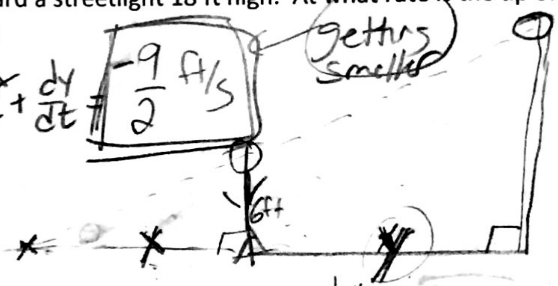
$2\frac{dx}{dt} = \frac{dy}{dt}$

$2\frac{dx}{dt} = 3$

$\frac{dx}{dt} = \frac{3}{2}\ ft/s$

$\frac{dx}{dt} + \frac{dy}{dt} = \boxed{-\frac{9}{2}\ ft/s}$

getting smaller



$\frac{dy}{dt} = -3\ ft/s$

3. A conical paper cup is 30 cm tall with a radius of 10 cm. The cup is being filled with water at a rate of 2π/3 cm³/sec. How fast is the water level rising when the water level is 2 cm?

$h = 2cm$

$\frac{dv}{dt} = \frac{2\pi}{3}$ $h = 2cm$

$\frac{dh}{dt} = $ ____

$V = \frac{1}{3}\pi r^2 h$

$V = \frac{1}{3}\pi\left(\frac{1}{3}h\right)^2 h$

$V = \frac{1}{3}\pi\left(\frac{h^2}{9}\right)h = \frac{h^3}{27}\pi$

$\frac{dv}{dt} = \frac{3\pi}{27}h^2\frac{dh}{dt}$

$\frac{2\pi}{3} = \frac{12\pi}{27}\frac{dh}{dt}$

$18\pi = 12\pi\frac{dh}{dt}$

$\boxed{\frac{dh}{dt} = \frac{3}{2}\ cm/s}$

$\frac{h}{r} = \frac{30}{10}$

$10h = 30r$

$r = \frac{1}{3}h$

4. An airplane is at an altitude of 5 miles and passes directly over a radar antenna. When the plane is 10 miles away, the radar detects that the distance s is changing at a rate of 240 mph. What is the speed of the airplane?

$z = 10$ $x =$

$\frac{dz}{dt} = 240$ $\frac{dx}{dt} = ?$

$x^2 + y^2 = z^2$

$2x\frac{dx}{dt} = 2z\frac{dz}{dt}$

$2(5\sqrt{5})\frac{dx}{dt} = 2(10)240$

$10\sqrt{5}\frac{dx}{dt} = 4{,}800$

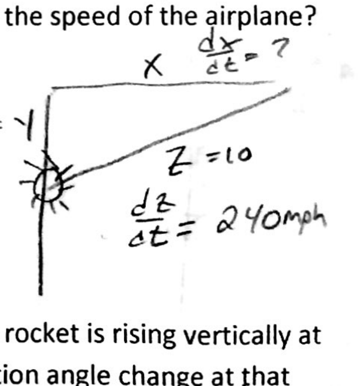
$\boxed{\frac{dx}{dt} = \frac{480}{\sqrt{5}}\ mph}$

$100 - 25 = x^2$

$x = \sqrt{75}$

$x = 5\sqrt{5}$

$5 = y$



5. A camera mounted at a point 3000 ft from the base of a rocket launching pad. If the rocket is rising vertically at 880 ft/s when it is 4000 ft above the launching pad, how fast must the camera elevation angle change at that instant to keep the camera aimed at the rocket?

$\tan\theta = \frac{y}{3000}$

$\sec^2\theta\frac{d\theta}{dt} = \frac{1}{3000}\frac{dy}{dt}$

$\sec^2(.927295)\frac{d\theta}{dt} = \frac{1}{3000}(880)$

$2.77777\ \frac{d\theta}{dt} = .293333$

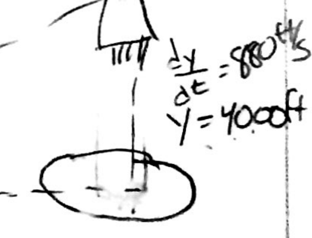
$\boxed{\frac{d\theta}{dt} = .1056\ Rad/s}$

$\tan\theta = \frac{4000}{3000}$

$\tan\theta = \frac{4}{3}$

$\theta = \tan^{-1}\left(\frac{4}{3}\right)$

$\theta = .927295$



1973 AB3/BC1

Given the curve $x + xy + 2y^2 = 6$.

(a) Find an expression for the slope of the curve at any point (x, y) on the curve.

(b) Write an equation for the line tangent to the curve at the point $(2,1)$.

(c) Find the coordinates of all other points on this curve with slope equal to the slope at $(2,1)$.

1970 AB4

A right circular cone and a hemisphere have the same base, and the cone is inscribed in the hemisphere. The figure is expanding in such a way that the combined surface area of the hemisphere and its base is increasing at a constant rate of 18 square inches per second. At what rate is the volume of the cone changing at the instant when the radius of the common base is 4 inches? Show your work.

<u>Note:</u> The surface area of a sphere of radius r is $S = 4\pi r^2$ and the volume of a right circular cone of height h and base radius r is $V = \frac{1}{3}\pi r^2 h$.

If $3x^2 - 2xy + y^2 = 2$, then the value of $\frac{dy}{dx}$ at $x = 1$ is

(A) -2 (B) 0 (C) 2 (D) 4 (E) not defined

When the area in square units of an expanding circle is increasing twice as fast as its radius in linear units, the radius is

(A) $\frac{1}{4\pi}$ (B) $\frac{1}{4}$ (C) $\frac{1}{\pi}$ (D) 1 (E) π

If $y = \tan u$, $u = v - \frac{1}{v}$, and $v = \ln x$, what is the value of $\frac{dy}{dx}$ at $x = e$?

(A) 0 (B) $\frac{1}{e}$ (C) 1 (D) $\frac{2}{e}$ (E) $\sec^2 e$

If $x^2 + xy + y^3 = 0$, then, in terms of x and y, $\frac{dy}{dx} =$

(A) $-\frac{2x+y}{x+3y^2}$ (B) $-\frac{x+3y^2}{2x+y}$ (C) $\frac{-2x}{1+3y^2}$ (D) $\frac{-2x}{x+3y^2}$ (E) $-\frac{2x+y}{x+3y^2-1}$

The volume of a cone of radius r and height h is given by $V = \frac{1}{3}\pi r^2 h$. If the radius and the height both increase at a constant rate of $\frac{1}{2}$ centimeter per second, at what rate, in cubic centimeters per second, is the volume increasing when the height is 9 centimeters and the radius is 6 centimeters?

(A) $\frac{1}{2}\pi$ (B) 10π (C) 24π (D) 54π (E) 108π

1973 AB3/BC1

a) $x + xy + 2y^2 = 6$

$1 + x\frac{dy}{dx} + y + 4y\frac{dy}{dx} = 0$

$x\frac{dy}{dx} + 4y\frac{dy}{dx} = -y - 1$

$\frac{dy}{dx}(x + 4y) = -y - 1$

$$\boxed{\frac{dy}{dx} = \frac{-y-1}{x+4y}}$$

b) $\boxed{y - 1 = \frac{-1}{3}(x - 2)}$

$\left.\frac{dy}{dx}\right|_{(2,1)} = \frac{-1-1}{2+4(1)} = \frac{-2}{6} = \frac{-1}{3}$

c) $\frac{-y-1}{x+4y} = \frac{-1}{3}$

$-1(x + 4y) = 3(-y - 1)$

$-x - 4y = -3y - 3$

$-x = y - 3$

$x = -y + 3$

$-y + 3 + (-y + 3)y + 2y^2 = 6$

$-y + 3 - y^2 + 3y + 2y^2 = 6$

$y^2 + 2y + 3 = 6$

$y^2 + 2y - 3 = 0$

$(y + 3)(y - 1) = 0$

$y = -3 \quad y = 1$

$$\boxed{(6, -3) \quad (2, 1)}$$

$x = -y + 3$

$x = -(-3) + 3$

$x = 3 + 3$

$x = 6$

$x = -y + 3$

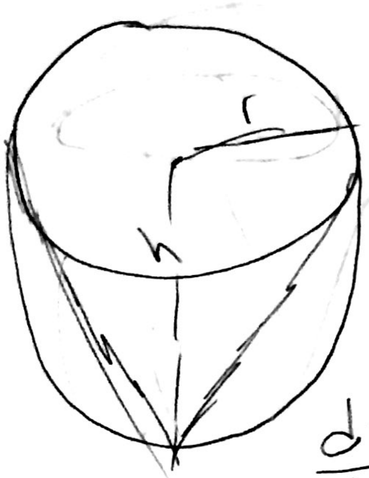
$x = -1 + 3$

$x = 2$

$x = 2$

1970 AB4

SA sphere: $4\pi r^2$



$r = h$

$\frac{ds}{dt} = 18 \text{ in}^2/\text{s}$

$\frac{dv}{dt} = \_\_\_$

$r = 4$

Surface Area of sphere and base

a) $SA = \frac{1}{2}(4\pi r^2) + \pi r^2$

$= 2\pi r^2 + \pi r^2$

$SA = 3\pi r^2$

$\frac{ds}{dt} = 6\pi r \frac{dr}{dt}$

$18 = 6\pi(4)\frac{dr}{dt}$

$3 = 4\pi \frac{dr}{dt}$

$\boxed{\frac{3}{4\pi} \text{ in/s} = \frac{dr}{dt}}$

$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi r^2 (r)$

$V = \frac{1}{3}\pi r^3$

$\frac{dv}{dt} = \pi r^2 \frac{dr}{dt}$

$\frac{dv}{dt} = \pi (4)^2 \frac{3}{4\pi} = \boxed{12 \text{ in}^3/\text{s}}$

MC #1) $3x^2+2xy+y^2=2 \quad @\, x=1$

$$6x+2x\frac{dy}{dx}+y(2)+2y\frac{dy}{dx}=0$$

$$2x\frac{dy}{dx}+2y\frac{dy}{dx}=-6x-2y$$

$$\frac{dy}{dx}(2x+2y)=-6x-2y$$

$$\frac{dy}{dx}=\frac{-3x-y}{x+y}$$

$$\left.\frac{dy}{dx}\right|_{(1,-1)} \frac{-3(1)-(-1)}{1+-1}=\frac{\quad}{0}$$

E

$$3(1)^2+2(1)y+y^2=2$$

$$3+2y+y^2=2 \qquad (1,-1)$$

$$y^2+2y+1=0$$

$$(y+1)(y+1)$$

$$y=-1$$

MC #2

$$A=\pi r^2$$

$$\frac{dA}{dt}=2\pi r\frac{dr}{dt}$$

$$2=2\pi r(1)$$

$$1=\pi r$$

$$\boxed{r=\frac{1}{\pi}} \qquad \boxed{C}$$

MC #3 $y = \tan u \quad u = v - \frac{1}{v} \quad v = \ln x$

What is the value of $\frac{dy}{dx}$ at $x = e$?

$$y = \tan\left(\ln x - \frac{1}{\ln x}\right)$$

$$\frac{dy}{dx} = \sec^2\left(\ln x - \frac{1}{\ln x}\right)\left(\frac{1}{x} - \frac{-\frac{1}{x}}{(\ln x)^2}\right)$$

$$\left.\frac{dy}{dx}\right|_{x=e} = \sec^2\left(\ln e - \frac{1}{\ln e}\right)\left(\frac{1}{e} - \frac{-\frac{1}{e}}{(\ln e)^2}\right)$$

$$= \sec^2(1-1)\left(\frac{1}{e} - -\frac{1}{e}\right)$$

$$= \frac{1}{(\cos(0))^2}\left(\frac{2}{e}\right)$$

$$= 1\left(\frac{2}{e}\right) = \boxed{\frac{2}{e}} \quad \boxed{D}$$

MC #4 If $x^2 + xy + y^3 = 0$

$$2x + x\frac{dy}{dx} + y + 3y^2\frac{dy}{dx} = 0$$

$$\frac{dy}{dx}(x + 3y^2) = -2x - y$$

$$\boxed{\frac{dy}{dx} = \frac{-2x-y}{x+3y^2} = \frac{-(2x+y)}{x+3y^2}} = \boxed{A}$$

MC#5

$$V = \frac{1}{3}\pi r^2 h$$

$$\frac{dv}{dt} = \frac{2}{3}\pi r \frac{dr}{dt} h + \frac{1}{3}\pi r^2 \frac{dh}{dt}$$

$$\frac{dv}{dt} = \frac{2}{3}\pi(6)\left(\frac{1}{2}\right)(9) + \frac{1}{3}\pi(6)^2\left(\frac{1}{2}\right)$$

$$\frac{dv}{dt} = 4\pi\left(\frac{1}{2}\right)(9) + 12\pi\left(\frac{1}{2}\right)$$

$$\frac{dv}{dt} = 18\pi + 6\pi = \boxed{24\pi \text{ cm}^3/\text{s}}$$

$\boxed{C}$