Parametric and Polar Free Response Answers:

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#1 1. (A) $v(0) = \langle 2, 5 \rangle$; Speed $= |v| = \sqrt{2^2 + 5^2} = \sqrt{29}$; $a = \left\langle \frac{d^2x}{dt^2}, \frac{d^2y}{dt^2} \right\rangle$; By nDeriv,

$a = \langle 0, 1 \rangle$.

(B) Since $V(0) = \langle 2, 5 \rangle, \frac{dy}{dx} = \frac{5}{2}$ $\quad y - 4 = \frac{2}{5}(x - 3)$

(C) $D = \int_0^2 \sqrt{(t^2 + 4) + (3e^t + 2e^{-t})^2}\, dt \approx 21.455$

(D) $x = 3 + \int_0^2 \sqrt{t^2 + 4}\, dt = 7.591$

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6. (A) $\frac{dx}{dt} = (t + 1)^{-1/2}$

$x = 2(t + 1)^{1/2} + C$

Initial conditions:

$-1 = 2\sqrt{4} + C$

$C = -5$

$x = 2\sqrt{t + 1} - 5$

(B) $\frac{dy}{dt} = 2x\frac{dx}{dt} - 2\frac{dx}{dt}$

$$\frac{dy}{dt} = \frac{2\left(2\sqrt{t + 1} - 5\right)}{\sqrt{t + 1}} - \frac{2}{\sqrt{t + 1}}$$

$$\frac{dy}{dt} = \frac{4\sqrt{t + 1} - 12}{\sqrt{t + 1}}$$

(C) Location: $(1, -1)$

$\frac{dx}{dt} = \frac{1}{3}$

$\frac{dy}{dt} = 0$

Speed $= \sqrt{\left(\frac{1}{3}\right)^2 + 0^2} = \frac{1}{3}$

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4. a. $-\tan t$ b. The slopes of the curved road and the two highways are the same at both A and B

c. $L = \int_0^{\frac{\pi}{2}} \sqrt{(-6\sin^2 t\cos t)^2 + (6\cos^2 t\sin t)^2}\,dt$

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2. a.

b. $A\langle -\sin t, 1\rangle$ $B\langle \cos t, 1\rangle$ c. A d. Both particles are moving upwards at the same rate.

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5. a. $\langle 2, -2t\rangle$ b. $y-(36-t^2)=(-t)(x-2t)$ or $y=-t(x-2t)+(36-t^2)$ c. $x=\frac{t^2+36}{t}$ d. 3

Page 12 #1

1. a. 40.172 b. 24.1644 c. $t = 1.4725079$ d. $y(3) = 20.0855$

AP Calculus BC

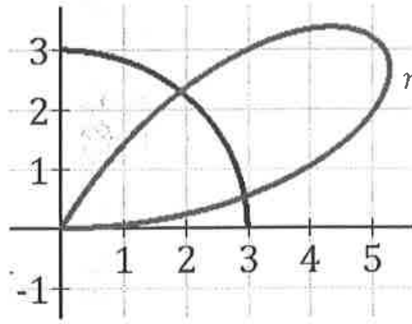
ANSWER KEY

CHAPTER 11 WORKSHEET

PARAMETRIC EQUATIONS AND POLAR COORDINATES

Derivatives and Equations in Polar Coordinates

1. The graphs of the polar curves $r_1 = 6\sin 3\theta$ and $r_2 = 3$ are shown to the right.
(You may use your calculator for all sections of this problem.)



a) Find the coordinates of the points of intersection of both curves for $0 \le \theta < \frac{\pi}{2}$. Write your answers using polar coordinates.

Points of intersection are collision points:

$$6\sin 3\theta = 3 \rightarrow \theta = \frac{\pi}{18} \text{ and } \frac{5\pi}{18}$$

Or $\theta \approx 0.1745$ and 0.8726

$r = 3 \rightarrow (3, 0.1745)$ and $(3, 0.8726)$

b) Write the coordinates of the points of intersection using now rectangular coordinates.

$$(3, 0.1745) \rightarrow \begin{cases} x = r\cdot\cos\theta = 2.954 \\ y = r\cdot\sin\theta = 0.5209 \end{cases} \rightarrow (2.954, 0.5209)$$

$$(3, 0.8726) \rightarrow \begin{cases} x = r\cdot\cos\theta = 1.928 \\ y = r\cdot\sin\theta = 2.298 \end{cases} \rightarrow (1.928, 2.298)$$

c) Find $\left.\frac{dr_1}{d\theta}\right|_{\theta=\frac{\pi}{4}}$. Interpret the meaning of your answer in the context of the problem.

By hand: $\frac{dr_1}{d\theta} = 18\cos 3\theta \rightarrow \left.\frac{dr_1}{d\theta}\right|_{\theta=\frac{\pi}{4}} = -9\sqrt{2}$

Using a calculator: $\left.\frac{d}{d\theta}(6\sin 3\theta)\right|_{\theta=\frac{\pi}{4}} \approx -12.7279$

When the graph of $r_1 = 6\sin 3\theta$ is traced at $\theta = \frac{\pi}{4}$ radians the distance to the pole is decreasing at a rate equal to 12.7279 units per radian.

d) For $0 \le \theta < \frac{\pi}{2}$, there are two points on r_1 with x-coordinate equal to 4. Find the subject points. Express your answer using polar coordinates.

$$x = r_1 \cdot \cos\theta = 6\sin 3\theta \cdot \cos\theta = 4 \rightarrow \theta \approx 0.253 \quad \text{and} \quad 0.696$$

$$\theta \approx 0.253 \rightarrow r_1 = 6\sin(3(0.253)) = 4.1317 \rightarrow (4.137, 0.253)$$

$$\theta \approx 0.696 \rightarrow r_1 = 6\sin(3(0.696)) = 5.213 \rightarrow (5.213, 0.696)$$

e) Write in terms of θ an expression for $\frac{dy}{dx}$, the slope of the tangent line to the graph of r_1.

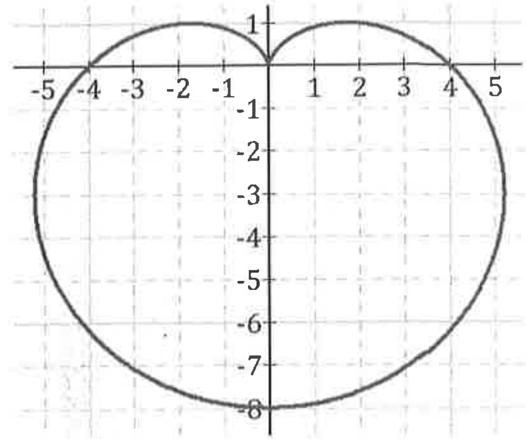
$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{3\cos 3\theta\sin\theta + \sin 3\theta\cos\theta}{3\cos 3\theta\cos\theta - \sin 3\theta\sin\theta}$$

f) Write in terms of x and y an equation for the line tangent to the graph of the curve r_1 at the point where $\theta = \frac{\pi}{4}$.

$$\left.\frac{dy}{dx}\right|_{\theta=\frac{\pi}{4}} = \frac{1}{2}$$

$$\left.\begin{matrix} x = r_1\cdot\cos\theta = 3 \\ y = r_1\cdot\sin\theta = 3 \end{matrix}\right\} \rightarrow y - 3 = \frac{1}{2}(x - 3)$$

2. The graph of the polar curve $r = 4 - 4\sin\theta$ is shown to the right.
(You may use your calculator for all sections of this problem.)



a) For $0 \le \theta < 2\pi$, there are two points on r with y-coordinate equal to -4. Find the subject points. Express your answers using polar coordinates.

$$y = r \cdot \sin\theta = (4 - 4\sin\theta)\sin\theta = -4$$
$$\to \theta \approx 3.8078 \quad \text{and} \quad 5.6169$$
$$\theta \approx 3.8078 \to r = 4 - 4\sin 3.8078 = 6.472$$
$$\to (6.472, 3.8078)$$
$$\theta \approx 5.6169 \to r = 4 - 4\sin 5.6169 = 6.472$$
$$\to (6.472, 5.6169)$$

b) Write an expression for the x-coordinate of each point on the graph of $r = 4 - 4\sin\theta$. Express your answer in terms of θ.

$$x = r \cdot \cos\theta = (4 - 4\sin\theta)\cos\theta$$

c) A particle moves along the polar curve $r = 4 - 4\sin\theta$ so that at time t seconds, $\theta = t^2$. Find the time t in the time interval $1 \le t \le 2$ for which the x-coordinate of the particle's position is -1.

$$x = (4 - 4\sin t^2)\cos t^2 = -1 \to t \approx 1.5536$$

d) Find $\left.\dfrac{dr}{dt}\right|_{t=2}$. Interpret the meaning of your answer in the context of the problem.

$$r = 4 - 4\sin t^2$$
$$\text{By hand: } \frac{dr}{dt} = -8t\cos t^2 \to \left.\frac{dr}{dt}\right|_{t=2} = -16\cos 4$$
$$\text{Using a calculator: } \left.\frac{d}{dt}(4 - 4\sin t^2)\right|_{t=2} \approx 10.458$$

As the particle moves on the graph of $r = 4 - 4\sin\theta$, when $t = 2$ seconds the distance to the pole is increasing at a rate equal to 10.458 units per second.

e) Find $\left.\dfrac{dx}{dt}\right|_{t=2}$. Interpret the meaning of your answer in the context of the problem.

$$\text{Using a calculator: } \left.\frac{d}{dt}((4 - 4\sin t^2)\cos t^2)\right|_{t=2} \approx 14.4368$$

As the particle moves on the graph of $r = 4 - 4\sin\theta$, when $t = 2$ seconds the particle moves to the right with a horizontal speed equal to 14.4368 units per second.

3. $r = \sqrt{2} + 2\sin\theta = 0 \Rightarrow \theta = \frac{5\pi}{4}; \frac{7\pi}{4}$

a) Since $\frac{dr}{d\theta} > 0$, r is increasing. This means that the curve is getting away from the origin.

b) $y = r\sin\theta \Rightarrow 3 = \left(\sqrt{2} + 2\sin\theta\right) \cdot \sin\theta$. Using the calculator: $\theta \approx 1.171$ or 1.970

c) $\frac{1}{2}\int_{5\pi/4}^{7\pi/4} \left(\sqrt{2} + 2\sin\theta\right)^2 d\theta \approx 0.142$

d) $\frac{dr}{d\theta} = 2\cos\theta \Rightarrow \int_{5\pi/4}^{7\pi/4} \sqrt{\left(\sqrt{2} + 2\sin\theta\right)^2 + (2\cos\theta)^2}\, d\theta \approx 1.440$

e) $\frac{1}{2}\int_{7\pi/4}^{13\pi/4} \left(\sqrt{2} + 2\sin\theta\right)^2 d\theta \approx 12.425$ or $\frac{1}{2}\int_{0}^{5\pi/4} \left(\sqrt{2} + 2\sin\theta\right)^2 d\theta + \frac{1}{2}\int_{7\pi/4}^{2\pi} \left(\sqrt{2} + 2\sin\theta\right)^2 d\theta \approx 12.425$

or $2 \cdot \left[\frac{1}{2}\int_{\pi/2}^{5\pi/4} \left(\sqrt{2} + 2\sin\theta\right)^2 d\theta\right] \approx 12.425$

f) $\int_{7\pi/4}^{13\pi/4} \sqrt{\left(\sqrt{2} + 2\sin\theta\right)^2 + (2\cos\theta)^2}\, d\theta \approx 12.754$ or

$\int_{0}^{5\pi/4} \sqrt{\left(\sqrt{2} + 2\sin\theta\right)^2 + (2\cos\theta)^2}\, d\theta + \int_{7\pi/4}^{2\pi} \sqrt{\left(\sqrt{2} + 2\sin\theta\right)^2 + (2\cos\theta)^2}\, d\theta \approx 12.754$ or

$2 \cdot \left[\int_{\pi/2}^{5\pi/4} \sqrt{\left(\sqrt{2} + 2\sin\theta\right)^2 + (2\cos\theta)^2}\, d\theta\right] \approx 12.754$

g) 12.283

4. $r = 4\sin 2\theta = 2 \Rightarrow \theta = \frac{\pi}{12}; \frac{5\pi}{12}$

a) $\frac{1}{2}\int_{0}^{\pi/12} (4\sin 2\theta)^2 d\theta + \frac{1}{2}\int_{\pi/12}^{5\pi/12} (2)^2 d\theta + \frac{1}{2}\int_{5\pi/12}^{\pi/2} (4\sin 2\theta)^2 d\theta \approx 2.457$ or

$2 \cdot \left[\frac{1}{2}\int_{0}^{\pi/12} (4\sin 2\theta)^2 d\theta\right] + \frac{1}{2}\int_{\pi/12}^{5\pi/12} (2)^2 d\theta \approx 2.457$

b) 2.457 × 4 =9.827

c) $\frac{dr}{d\theta} = 8\cos 2\theta$ and $\frac{dr}{d\theta} = 0$

$\int_{0}^{\pi/12} \sqrt{(4\sin 2\theta)^2 + (8\cos 2\theta)^2}\, d\theta + \int_{\pi/12}^{5\pi/12} \sqrt{(2)^2 + (0)^2}\, d\theta + \int_{5\pi/12}^{\pi/2} \sqrt{(4\sin 2\theta)^2 + (8\cos 2\theta)^2}\, d\theta \approx 6.143$

or $2 \cdot \left[\int_{0}^{\pi/12} \sqrt{(4\sin 2\theta)^2 + (8\cos 2\theta)^2}\, d\theta\right] + \int_{\pi/12}^{5\pi/12} \sqrt{(2)^2 + (0)^2}\, d\theta \approx 6.143$

5. a)

At P, $\frac{5}{3}y = \sqrt{1+y^2}$, so $y = \frac{3}{4}$.

Since $x = \frac{5}{3}y$, $x = \frac{5}{4}$.

$\frac{dx}{dy} = \frac{y}{\sqrt{1+y^2}} = \frac{y}{x}$. At P, $\frac{dx}{dy} = \frac{3/4}{5/4} = \frac{3}{5}$.

$2 : \begin{cases} 1 : \text{coordinates of } P \\ 1 : \frac{dx}{dy} \text{ at } P \end{cases}$

b)

$x = r\cos\theta\ ;\ y = r\sin\theta$

$x^2 - y^2 = 1 \Rightarrow r^2\cos^2\theta - r^2\sin^2\theta = 1$

$r^2 = \frac{1}{\cos^2\theta - \sin^2\theta}$

$2 : \begin{cases} 1 : \text{substitutes } x = r\cos\theta \text{ and } y = r\sin\theta \text{ into } x^2 - y^2 = 1 \\ 1 : \text{isolates } r^2 \end{cases}$

c)

Let β be the angle that segment OP makes with the x-axis. Then $\tan\beta = \frac{y}{x} = \frac{3/4}{5/4} = \frac{3}{5}$.

$$\text{Area} = \int_0^{\tan^{-1}(3/5)} \frac{1}{2}r^2\, d\theta$$

$$= \frac{1}{2}\int_0^{\tan^{-1}(3/5)} \frac{1}{\cos^2\theta - \sin^2\theta}\, d\theta$$

$2 : \begin{cases} 1 : \text{limits} \\ 1 : \text{integrand and constant} \end{cases}$