8.7 L'Hôpital's Rule

If $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)}$ is : $\dfrac{0}{0}$ or $\dfrac{\infty}{\infty}$

then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

1) $\lim\limits_{x\to 0} \dfrac{\sin x}{x}$ $\quad \dfrac{0}{0}$

$\dfrac{\cos x}{1} = \dfrac{1}{1} = \boxed{1}$

2) $\lim\limits_{x\to 0} \dfrac{3x-\sin x}{x}$ $\quad \dfrac{0}{0}$

$\dfrac{3-\cos x}{1} = \dfrac{2}{1} = \boxed{2}$

3) $\lim\limits_{x\to 0} \dfrac{\sqrt{1+x}-1}{x}$ $\quad \dfrac{0}{0}$

$\dfrac{\frac{1}{2}(1+x)^{-1/2}}{1} = \dfrac{\frac{1}{2}}{1} = \boxed{\frac{1}{2}}$

4) $\lim\limits_{x\to 0} \dfrac{x-\sin x}{x^3}$ $\quad \dfrac{0}{0}$

$\dfrac{1-\cos x}{3x^2} = \dfrac{0}{0}$

$\dfrac{\sin x}{6x} = \dfrac{0}{0}$

$\dfrac{\cos x}{6} = \boxed{\frac{1}{6}}$

5) $\lim\limits_{x\to 0} \dfrac{\sqrt{1+x}-1-\frac{x}{2}}{x^2}$ $\quad \dfrac{0}{0}$

$\dfrac{\frac{1}{2}(1+x)^{-1/2}-\frac{1}{2}}{2x} = \dfrac{0}{0}$

$\dfrac{-\frac{1}{4}(1+x)^{-3/2}}{2} = \dfrac{-\frac{1}{4}}{2} = \boxed{-\frac{1}{8}}$

6) $\lim_{x\to\infty} \frac{\ln x}{x} = 0$ $\quad x \to$ big

$\frac{\infty}{\infty}$

$$\frac{\frac{1}{x}}{1} = \frac{\frac{1}{\infty}}{1} = \boxed{0}$$

7) $\lim_{x\to 1} \frac{x^2-1}{x^2-x}$ $\quad \frac{0}{0}$

$$\frac{2x}{2x-1} = \frac{2}{1} = \boxed{2}$$

MORE:

1) $\lim_{x\to 0} \frac{\cos x - 1}{x}$ $\quad \frac{0}{0}$

$$\frac{\sin x}{1} = \boxed{0}$$

2) $\lim_{h\to 0} \frac{\sqrt[3]{8+h} - 2}{h}$ $\quad \frac{0}{0}$

$$\frac{\frac{1}{3}(8+h)^{-2/3} - 2}{1} = \frac{\frac{1}{3}\cdot\frac{1}{4} - 2}{1}$$

$$= \boxed{\frac{-23}{12}}$$

3) $\lim_{x\to 2} \frac{x^2-4}{x^2+4} = \frac{0}{8} = \boxed{0}$

4) $\lim_{x\to\infty} \sin x = \boxed{\text{DNE}}$

5) $\lim_{x\to\infty} \frac{e^x}{x^{50}}$ $\quad \frac{\infty}{\infty}$

$$\frac{e^x}{50x^{49}}$$

$\vdots$

$$\frac{e^x}{c} = \boxed{\infty}$$

6) $\lim_{x\to\infty} \cos\left(\frac{1}{x}\right) = \boxed{1}$

More L'Hopital's Day 1 (Warm-Up Day 2)

1) $\lim_{x \to 1} \frac{5 \ln x}{x-1}$ $\frac{0}{0}$

$\frac{5 \cdot \frac{1}{x}}{1} = 5 \cdot 1 = \boxed{5}$

2) $\lim_{x \to 0} \frac{3x}{\ln(x+1)}$ $\frac{0}{0}$

$\frac{3}{\frac{1}{x+1}} = 3(x+1) = \boxed{3}$

3) $\lim_{x \to 0} \frac{e^x - e^{-x}}{x}$ $\frac{0}{0}$

$\frac{e^x + e^x}{1} = \frac{1+1}{1} = \boxed{2}$

4) $\lim_{x \to \infty} \frac{e^x}{x^2}$ $\frac{\infty}{\infty}$

$\frac{e^x}{2x}$ $\frac{\infty}{\infty}$

$\frac{e^x}{2} = \boxed{\infty}$

5) $\lim_{x \to \infty} \frac{x^3}{e^{3x}}$ $\frac{\infty}{\infty}$

$\frac{\cancel{3}x^2}{\cancel{3}e^{3x}} = \frac{x^2}{e^{3x}}$ $\frac{\infty}{\infty}$

$\frac{2x}{3e^{3x}}$ $\frac{\infty}{\infty}$

$\frac{2}{9e^{3x}} = \boxed{0}$

6) $\lim_{x \to \infty} \frac{\ln x}{x^2}$ $\frac{\infty}{\infty}$

$\frac{\frac{1}{x}}{2x} = \frac{1}{x} \cdot \frac{1}{2x} = \frac{1}{2x^2}$

$= \boxed{0}$

8.7 L'H Day 2

Other indeterminate forms:

$\frac{\infty}{\infty}$ $\quad \infty \cdot 0 \quad \infty - \infty \quad 1^{\infty} \quad 0^0 \quad \infty^0$

Why?

$\lim_{x \to 0} x\left(\frac{1}{x}\right) \quad \lim_{x \to 0} x\left(\frac{2}{x}\right) \quad \lim_{x \to \infty} x\left(\frac{1}{e^x}\right) \quad \lim_{x \to \infty} e^x\left(\frac{1}{x}\right)$

$= 1 \qquad = 2 \qquad = 0 \qquad = \infty$

All give $0 \cdot \infty$, but have dif limits

$\therefore$ indeterminate!

* have to change indeterminate forms to fractions first

1) $\lim_{x \to \infty} \left(x \sin \frac{1}{x}\right) \qquad \infty \cdot 0$

#1

$\lim_{x \to \infty} \frac{\sin \frac{1}{x}}{\frac{1}{x}} \qquad \frac{0}{0}$

we Know $\lim_{x \to 0} \frac{\sin x}{x} = \boxed{1}$

L'H:

$$\lim_{x \to \infty} \frac{\sin \frac{1}{x}}{\frac{1}{x}} = \lim_{x \to \infty} \frac{\left(\cos \frac{1}{x}\right)\left(-x^{-2}\right)}{-x^{-2}} = \lim_{x \to \infty} \frac{\left(\cos \frac{1}{x}\right)\cancel{\left(-\frac{1}{x^2}\right)}}{\cancel{\left(\frac{-1}{x^2}\right)}}$$

$$= \lim_{x \to \infty} \cos\left(\frac{1}{x}\right) = \cos 0 = \boxed{1}$$

2) $\lim_{x \to 1} \left(\frac{1}{\ln x} - \frac{1}{x-1} \right)$ $\quad \infty - \infty$

#3

$\lim_{x \to 1} \frac{x-1}{\ln x(x-1)} - \frac{\ln x}{\ln x(x-1)}$

$\lim_{x \to 1} \frac{x-1-\ln x}{\ln x(x-1)} \quad \frac{0}{0}$

L'H : $\lim_{x \to 1} \frac{1 - \frac{1}{x}}{\ln x \cdot 1 + (x-1) \cdot \frac{1}{x}}$

$\lim_{x \to 1} \frac{x-1}{x\ln x + (x-1)} \quad \frac{0}{0}$

L'H : $\lim_{x \to 1} \frac{1}{x \cdot \frac{1}{x} + \ln x \cdot 1 + 1}$

$\lim_{x \to 1} \frac{1}{1 + \ln x + 1} = \boxed{\frac{1}{2}}$

3) $\lim_{x \to \infty} x^{1/x} \quad \infty^0$

#4

$y = \lim_{x \to \infty} x^{1/x}$

$\ln y = \lim_{x \to \infty} \ln x^{1/x}$

$= \lim_{x \to \infty} \frac{1}{x} \cdot \ln x = \frac{\ln x}{x}$

L'H : $\lim_{x \to \infty} \frac{\frac{1}{x}}{1} = 0$

$\ln y = 0$

$e^0 = y$

$\boxed{y = 1}$

4) $\lim_{x \to \infty} \left(\sqrt{x^2+x} - x\right)$ $\quad \infty - \infty$

$$= \frac{\sqrt{x^2+x} - x}{1} \cdot \frac{\sqrt{x^2+x} + x}{\sqrt{x^2+x} + x}$$

$$= \frac{x^2+x-x^2}{\sqrt{x^2+x}+x} = \frac{x}{\sqrt{x^2+x}+x} \quad \frac{\infty}{\infty}$$

L'H: $\dfrac{1}{\frac{1}{2}(x^2+x)^{-1/2}(2x+1)+1} = \dfrac{1}{\frac{2x+1}{2\sqrt{x^2+x}}+1}$

$$\frac{1}{\frac{2x}{2x}+1} = \frac{1}{1+1} = \boxed{\frac{1}{2}} \qquad \frac{x}{x+x} = \frac{x}{2x} = \boxed{\frac{1}{2}}$$

5) $\lim_{x \to 0^+} (4x+1)^{\cot x}$ $\quad 1^\infty$

#5

$y = \lim_{x \to 0^+} (4x+1)^{\cot x}$

$\ln y = \lim_{x \to 0^+} \ln(4x+1)^{\cot x}$

$\ln y = \lim_{x \to 0^+} \cot x \ln(4x+1)$

$$\frac{\ln(4x+1)}{\tan x} \quad \frac{0}{0}$$

L'H: $\dfrac{\frac{1}{4x+1} \cdot 4}{\sec^2 x} = \dfrac{4}{1} = 4$

$\ln y = 4$

$y = \boxed{e^4}$

6) $\lim_{x\to\infty} x \tan\frac{1}{x}$ $\quad \infty \cdot 0$

#2

$$\frac{\tan\frac{1}{x}}{\frac{1}{x}} \quad \frac{0}{0}$$

$$L'H: \frac{\sec^2\left(\frac{1}{x}\right)\cdot\cancel{\left(-\frac{1}{x^2}\right)}}{\cancel{\left(-\frac{1}{x^2}\right)}} = \sec^2\left(\frac{1}{x}\right) = \boxed{1}$$

#6

7) $\lim_{x\to\infty} \left(1+\frac{1}{x}\right)^x$ $\quad 1^\infty$

$$y = \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$$

$$\ln y = \lim_{x\to\infty} \ln\left(1+\frac{1}{x}\right)^x$$

$$\ln y = \lim_{x\to\infty} x\ln\left(1+\frac{1}{x}\right)$$

$$\lim_{x\to\infty}\frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}} \quad \frac{0}{0}$$

$$L'H: \lim_{x\to\infty}\frac{\frac{1}{1+\frac{1}{x}}\cdot\cancel{\left(-\frac{1}{x^2}\right)}}{\cancel{\left(-\frac{1}{x^2}\right)}}$$

$$\lim_{x\to\infty}\frac{1}{1+\frac{1}{x}} = \frac{1}{1} = 1$$

$$\ln y = 1$$

$$y = e^1 = \boxed{e}$$

#7

8) $\lim_{x\to 0^+} (\sin x)^x$ $\quad 0^0$

$$y = \lim_{x\to 0^+}(\sin x)^x$$

$$\ln y = \lim_{x\to 0^+} \ln(\sin x)^x$$

$$\ln y = \lim_{x\to 0^+} x\ln(\sin x)$$

$$\lim_{x\to 0^+}\frac{\ln(\sin x)}{\frac{1}{x}} \quad \frac{-\infty}{\infty}$$

$$L'H: \frac{\frac{1}{\sin x}\cdot\cos x}{\frac{-1}{x^2}} = \frac{\cot x}{-x^{-2}} = \frac{-x^2}{\tan x}$$

$$\lim_{x\to 0^+}\frac{-x^2}{\tan x} \quad \frac{0}{0}$$

$$L'H: \lim_{x\to 0^+}\frac{-2x}{\sec^2 x} = \frac{0}{1} = 0$$

$$\ln y = 0 \quad y = e^0 = \boxed{1}$$