AP Calculus BC

Taylor's Theorem and Lagrange Error Bound

Name: Key

<u>Lagrange Form of the Remainder (also called Lagrange Error Bound or Taylor's Theorem Remainder)</u>

When a Taylor polynomial is used to approximate a function, we need a way to see how accurately the polynomial approximates the function.

$f(x)=P_n(x)+R_n(x)$ so $R_n(x)=f(x)-P_n(x)$

Written in words:

Function = Polynomial + Remainder so Remainder = Function - Polynomial

<u>Taylor's Theorem</u>: If a function f is differentiable through order n + 1 in an interval containing c, then for each x in the interval, there exists a number z between x and c such that

$$f(x)=f(c)+f'(c)(x-c)+\frac{f''(c)}{2!}(x-c)^2+\ldots+\frac{f^{(n)}(x)}{n!}(x-c)^n+R_n(x)$$

where the remainder $R_n(x)$ (or error) is given by $R_n(x)=\frac{f^{(n+1)}(z)}{(n+1)!}(x-c)^{n+1}$ **(The Lagrange Remainder)**

$f^{(n+1)}$ = (n+1) derivative

z = # that maximizes (n+1) derivative between x and c

x = estimate

c = center

n = degree

<u>Example 1:</u> Find the upper bound for error when approximating e^2 with a <u>MacLaurin</u> polynomial with degree 8.

$$|R_8(x)| \le \left|\frac{f^9(x)\cdot(x-c)^9}{9!}\right|$$

$0 \le z \le 2$

$\downarrow$

max @ e^2

$f^9(x)=e^x$

$$\frac{e^2(2-0)^9}{9!} = .010425$$

<u>Example 2:</u> Using the formula for the error E, what is the maximum value of the error in approximating ln 1.2 with a Taylor polynomial of degree 3 centered at x =1?

$f(x)=\ln x$

$f'(x)=\frac{1}{x}=x^{-1}$

$f''(x)=-1x^{-2}$

$f'''(x)=2x^{-3}$

$f^{(4)}(x)=-6x^{-4}=\frac{-6}{x^4}$

$1 \le z \le 1.2$

max at $\left|\frac{-6}{1^4}\right|=6$

$$|\text{Error}| \le \left|\frac{6(1.2-1)^4}{4!}\right|$$

$|E| \le 4\times 10^{-4}$

$= .0004$

<u>Example 3:</u> Find the upper bound for error when approximating $e^{-1.5}$ with a 4^{th} degree Taylor polynomial with c = 0.

$f^5(x)=e^x$

$-1.5 \le z \le 0$

$\downarrow$

max at $e^0=1$

$$|\text{Error}| \le \left|\frac{e^0(-1.5-0)^5}{5!}\right| = .06328$$

Example 3:

Let f be the function given by $f(x) = cos\left(3x + \frac{\pi}{6}\right)$ and let P(x) be the fourth-degree Taylor polynomial for f about x = 0.

(a) Find P(x).

(b) Use the Lagrange error bound to show that $\left|f\left(\frac{1}{6}\right) - P\left(\frac{1}{6}\right)\right| < \frac{1}{3000}$

a) $f(x) = \cos(3x + \frac{\pi}{6})$ $f(0) = \sqrt{3}/2$

$f'(x) = -3\sin(3x + \frac{\pi}{6})$ $f'(0) = -\frac{3}{2}$

$f''(x) = -9\cos(3x + \frac{\pi}{6})$ $f''(0) = -\frac{9\sqrt{3}}{2}$

$f'''(x) = 27\sin(3x + \frac{\pi}{6})$ $f'''(0) = \frac{27}{2}$

$f^{(4)}(x) = 81\cos(3x + \frac{\pi}{6})$ $f^{4}(0) = \frac{81\sqrt{3}}{2}$

$$P_4(x) = \frac{\sqrt{3}}{2} - \frac{3x}{2} - \frac{9\sqrt{3}x^2}{2 \cdot 2!} + \frac{27x^3}{2 \cdot 3!} + \frac{81\sqrt{3}x^4}{2 \cdot 4!}$$

b) $\underbrace{\left|f\left(\frac{1}{6}\right) - P\left(\frac{1}{6}\right)\right|}_{\text{Find error!}} < \frac{1}{3000}$

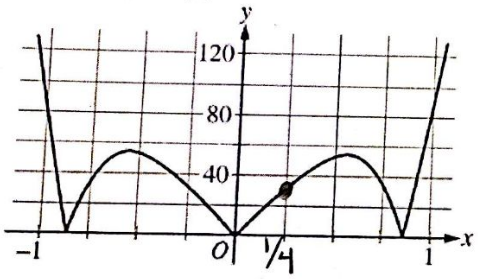
$$R_4(x) < \left|\frac{f^5(z)(x-0)^5}{5!}\right|$$

$f^5(x) = -243\sin(3x + \frac{\pi}{6})$ ↳ max at 243

$$R_4(x) < \frac{243\left(\frac{1}{6}\right)^5}{5!} = \frac{1}{32 \cdot 5!} = \frac{1}{3840} < \frac{1}{3000}$$

Example 4:

2011 BC6, part (d)



Graph of $y = \left|f^{(5)}(x)\right|$

Let $f(x) = \sin(x^2) + \cos(x)$. The graph of $y = \left|f^{(5)}(x)\right|$ is shown above.

...

(d) Let $P_4(x)$ be the fourth-degree Taylor polynomial for f about $x = 0$. Using the information from the graph of $y = \left|f^{(5)}(x)\right|$ shown above, show that $\underbrace{\left|P_4\left(\frac{1}{4}\right) - f\left(\frac{1}{4}\right)\right|}_{\text{Find error!}} < \frac{1}{3000}$

$$R_4(x) < \left|\frac{f^5(z) \cdot x^5}{5!}\right|$$

$$\frac{40x^5}{5!} \Rightarrow \frac{40\left(\frac{1}{4}\right)^5}{5!} = \frac{5}{128 \cdot 5!} = \frac{1}{3072} < \frac{1}{3000}$$