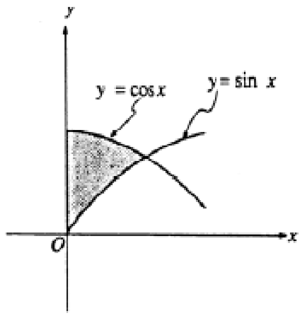
1991 BC3



Let R be the shaded region in the first quadrant enclosed by the y-axis and the graphs of $y = \sin x$ and $y = \cos x$, as shown in the figure above.

(a) Find the area of R .

(b) Find the volume of the solid generated when R is revolved about the x-axis.

(c) Find the volume of the solid whose base is R and whose cross sections cut by planes perpendicular to the x-axis are squares.

1991 BC3
Solution

(a)
$$\begin{aligned}\text{Area} &= \int_0^{\pi/4} \cos x - \sin x \, dx \\ &= \left(\sin x + \cos x\right)\Big|_0^{\pi/4} \\ &= \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right) - (0+1) \\ &= \sqrt{2} - 1\end{aligned}$$

(b)
$$\begin{aligned}V &= \pi \int_0^{\pi/4} \cos^2 x - \sin^2 x \, dx \\ &= \pi \int_0^{\pi/4} \cos 2x \, dx \\ &= \frac{\pi}{2} \sin 2x \Big|_0^{\pi/4} \\ &= \frac{\pi}{2}(1-0) = \frac{\pi}{2}\end{aligned}$$

(c)
$$\begin{aligned}V &= \int_0^{\pi/4} \left(\cos x - \sin x\right)^2 dx \\ &= \int_0^{\pi/4} 1 - 2\sin x \cos x \, dx \\ &= \left(x - \sin^2 x\right)\Big|_0^{\pi/4} \\ &= \frac{\pi}{4} - \frac{1}{2} - (0-0) \\ &= \frac{\pi}{4} - \frac{1}{2}\end{aligned}$$

1991 BC6

A certain rumor spreads through a community at the rate $\frac{dy}{dt} = 2y(1-y)$, where y is the proportion of the population that has heard the rumor at time t.

(a) What proportion of the population has heard the rumor when it is spreading the fastest?

(b) If at time $t = 0$ ten percent of the people have heard the rumor, find y as a function of t.

(c) At what time t is the rumor spreading the fastest?

1991 BC6
Solution

(a) $2y(1-y) = 2y - 2y^2$ is largest when $2 - 4y = 0$

so proportion is $y = \frac{1}{2}$

(b) $\frac{1}{y(1-y)}dy = 2\,dt$

$\int \frac{1}{y(1-y)}dy = \int 2\,dt$

$\int \frac{1}{y} + \frac{1}{1-y}dy = \int 2\,dt$

$\ln y - \ln(1-y) = 2t + C$

$\ln\frac{y}{1-y} = 2t + C$

$\frac{y}{1-y} = ke^{2t}$

$y(0) = 0.1 \Rightarrow k = \frac{1}{9}$

$y = \frac{e^{2t}}{9 + e^{2t}}$

(c) $\frac{\frac{1}{2}}{1-\frac{1}{2}} = \frac{1}{9}e^{2t}$

$1 = \frac{1}{9}e^{2t}$

$t = \frac{1}{2}\ln 9 = \ln 3$

1993 AB6

Let $P(t)$ represent the number of wolves in a population at time t years, when $t \geq 0$. The population $P(t)$ is increasing at a rate directly proportional to $800 - P(t)$, where the constant of proportionality is k.

(a) If $P(0) = 500$, find $P(t)$ in terms of t and k.

(b) If $P(2) = 700$, find k.

(c) Find $\lim_{t \to \infty} P(t)$.

1993 AB 6
Solution

(a) $P'(t) = k(800 - P(t))$

$$\frac{dP}{800 - P} = k\,dt$$

$$-\ln|800 - P| = kt + C_0$$

$$|800 - P| = C_1 e^{-kt}$$

$$800 - 500 = C_1 e^0$$

$$C_1 = 300$$

Therefore $P(t) = 800 - 300e^{-kt}$

(b) $P(2) = 700 = 800 - 300e^{-2k}$

$$k = \frac{\ln 3}{2} \simeq 0.549$$

(c) $\lim_{t \to \infty}\left(800 - 300e^{-\frac{\ln 3}{2}t}\right) = 800$

1996 AB2

Let R be the region in the first quadrant under the graph of $y = \dfrac{1}{\sqrt{x}}$ for $4 \le x \le 9$.

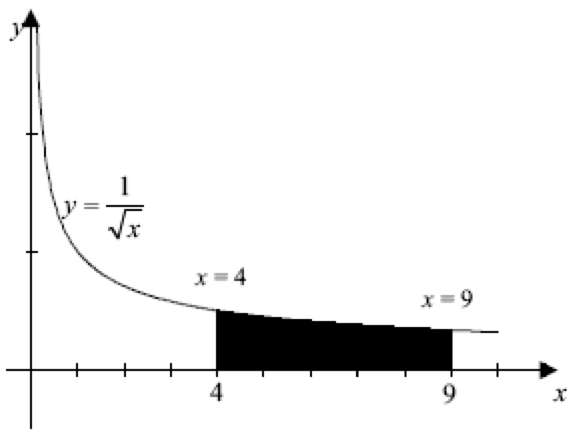
(a) Find the area of R.

(b) If the line $x = k$ divides the region R into two regions of equal area, what is the value of k?

(c) Find the volume of the solid whose base is the region R and whose cross sections cut by planes perpendicular to the x-axis are squares.

1996 AB2
Solution

(a) $\displaystyle\int_4^9 \frac{dx}{\sqrt{x}}dx = 2$



(b) $\displaystyle\int_4^k \frac{dx}{\sqrt{x}}dx = 1$

$2\sqrt{x}\Big]_4^k = 1$

$2\sqrt{k} - 2\sqrt{4} = 1$

$k = \dfrac{25}{4}$

$\left(\text{or } \displaystyle\int_k^9 \frac{dx}{\sqrt{x}} = 1 \text{ or } \int_4^k \frac{dx}{\sqrt{x}} = \int_k^9 \frac{dx}{\sqrt{x}}\right)$

(c) $\text{Volume} = \displaystyle\int_4^9 \left(\frac{1}{\sqrt{x}}\right)^2 dx$

$\displaystyle = \int_4^9 \frac{dx}{x} = \ln x\Big|_4^9 = \ln\frac{9}{4} \quad (\text{or } 0.811)$

1997 BC3

Let R be the region enclosed by the graphs of $y = \ln\left(x^2+1\right)$ and $y = \cos x$.

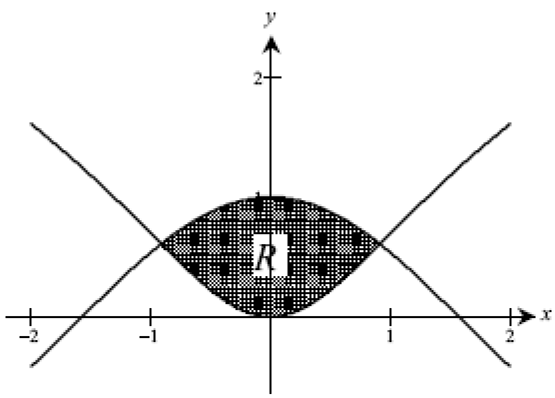
(a) Find the area of R .

(b) Write an expression involving one or more integrals that gives the length of the boundary of the region R . Do not evaluate.

(c) The base of a solid is the region R . Each cross section of the solid perpendicular to the x-axis is an equilateral triangle. Write an expression involving one or more integrals that gives the volume of the solid. Do not evaluate.

1997 BC3
Solution

(a)



$\ln\left(x^2+1\right) = \cos x$

$x = \pm 0.91586$

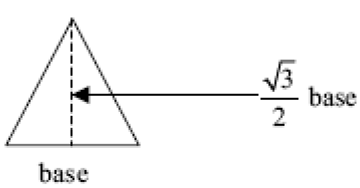
let $B = 0.91586$

area $= \int_{-B}^{B}\left[\cos x - \ln\left(x^2+1\right)\right]dx$

$= 1.168$

(b) $L = \int_{-B}^{B}\sqrt{1+\left(\frac{2x}{x^2+1}\right)^2}\,dx + \int_{-B}^{B}\sqrt{1+(-\sin x)^2}\,dx$

(c)



area of cross section $= \frac{1}{2}\left[\cos x - \ln\left(x^2+1\right)\right]\times\left[\frac{\sqrt{3}}{2}\left(\cos x - \ln\left(x^2+1\right)\right)\right]$

volume $= \int_{-B}^{B}\frac{\sqrt{3}}{4}\left[\cos x - \ln\left(x^2+1\right)\right]^2 dx$