

1.

Let f be the function given by $f(x) = e^{-2x^2}$.

- (a) Find the first four nonzero terms and the general term of the power series for $f(x)$ about $x=0$.
- (b) Find the interval of convergence of the power series for $f(x)$ about $x=0$. Show the analysis that leads to your conclusion.
- (c) Let g be the function given by the sum of the first four nonzero terms of the power series for $f(x)$ about $x=0$. Show that $|f(x) - g(x)| < 0.02$ for $-0.6 \leq x \leq 0.6$.

2. Let $P(x) = 7 - 3(x-4) + 5(x-4)^2 - 2(x-4)^3 + 6(x-4)^4$ be the fourth-degree Taylor polynomial for the function f about 4. Assume f has derivatives of all orders for all real numbers.

- (a) Find $f(4)$ and $f'''(4)$.
- (b) Write the second-degree Taylor polynomial for f' about 4 and use it to approximate $f'(4.3)$.
- (c) Write the fourth-degree Taylor polynomial for $g(x) = \int_4^x f(t) dt$ about 4.
- (d) Can $f(3)$ be determined from the information given? Justify your answer.

3. The Maclaurin series for $f(x)$ is given by $1 + \frac{x}{2!} + \frac{x^2}{3!} + \frac{x^3}{4!} + \cdots + \frac{x^n}{(n+1)!} + \cdots$
- Find $f'(0)$ and $f^{(17)}(0)$.
 - For what values of x does the given series converge? Show your reasoning.
 - Let $g(x) = x f(x)$. Write the Maclaurin series for $g(x)$, showing the first three nonzero terms and the general term.
 - Write $g(x)$ in terms of a familiar function without using series. Then, write $f(x)$ in terms of the same familiar function.
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4. Let f be a function that has derivatives of all orders for all real numbers. Assume $f(1) = 3$, $f'(1) = -2$, $f''(1) = 2$, and $f'''(1) = 4$.
- Write the second-degree Taylor polynomial for f about $x = 1$ and use it to approximate $f(0.7)$.
 - Write the third-degree Taylor polynomial for f about $x = 1$ and use it to approximate $f(1.2)$.
 - Write the second-degree Taylor polynomial for f' , the derivative of f , about $x = 1$ and use it to approximate $f'(1.2)$.
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5. Let f be the function defined by $f(x) = \frac{1}{x-1}$.
- Write the first four terms and the general term of the Taylor series expansion of $f(x)$ about $x = 2$.
 - Use the result from part (a) to find the first four terms and the general term of the series expansion about $x = 2$ for $\ln|x-1|$.
 - Use the series in part (b) to compute a number that differs from $\ln\frac{3}{2}$ by less than 0.05. Justify your answer.

The Maclaurin series for $\ln\left(\frac{1}{1-x}\right)$ is $\sum_{n=1}^{\infty} \frac{x^n}{n}$ with interval of convergence $-1 \leq x < 1$.

6. (a) Find the Maclaurin series for $\ln\left(\frac{1}{1+3x}\right)$ and determine the interval of convergence.
- (b)

Omit

- (c) Give a value of p such that $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^p}$ converges, but $\sum_{n=1}^{\infty} \frac{1}{n^{2p}}$ diverges. Give reasons why your value of p is correct.
- (d) Give a value of p such that $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges, but $\sum_{n=1}^{\infty} \frac{1}{n^{2p}}$ converges. Give reasons why your value of p is correct.

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7. The function f has derivatives of all orders for all real numbers x . Assume $f(2) = -3$, $f'(2) = 5$, $f''(2) = 3$, and $f'''(2) = -8$.
- (a) Write the third-degree Taylor polynomial for f about $x = 2$ and use it to approximate $f(1.5)$.
- (b) The fourth derivative of f satisfies the inequality $|f^{(4)}(x)| \leq 3$ for all x in the closed interval $[1.5, 2]$. Use the Lagrange error bound on the approximation to $f(1.5)$ found in part (a) to explain why $f(1.5) \neq -5$.
- (c) Write the fourth-degree Taylor polynomial, $P(x)$, for $g(x) = f(x^2 + 2)$ about $x = 0$. Use P to explain why g must have a relative minimum at $x = 0$.

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8. Let f be a function with derivatives of all orders and for which $f(2) = 7$. When n is odd, the n th derivative of f at $x = 2$ is 0. When n is even and $n \geq 2$, the n th derivative of f at $x = 2$ is given by $f^{(n)}(2) = \frac{(n-1)!}{3^n}$.
- (a) Write the sixth-degree Taylor polynomial for f about $x = 2$.
- (b) In the Taylor series for f about $x = 2$, what is the coefficient of $(x-2)^{2n}$ for $n \geq 1$?
- (c) Find the interval of convergence of the Taylor series for f about $x = 2$. Show the work that leads to your answer.

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9. The function f has a Taylor series about $x = 2$ that converges to $f(x)$ for all x in the interval of convergence. The n th derivative of f at $x = 2$ is given by $f^{(n)}(2) = \frac{(n+1)!}{3^n}$ for $n \geq 1$, and $f(2) = 1$.
- Write the first four terms and the general term of the Taylor series for f about $x = 2$.
 - Find the radius of convergence for the Taylor series for f about $x = 2$. Show the work that leads to your answer.
 - Let g be a function satisfying $g(2) = 3$ and $g'(x) = f(x)$ for all x . Write the first four terms and the general term of the Taylor series for g about $x = 2$.
 - Does the Taylor series for g as defined in part (c) converge at $x = -2$? Give a reason for your answer.
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10. A function f is defined by

$$f(x) = \frac{1}{3} + \frac{2}{3^2}x + \frac{3}{3^3}x^2 + \cdots + \frac{n+1}{3^{n+1}}x^n + \cdots$$

for all x in the interval of convergence of the given power series.

- Find the interval of convergence for this power series. Show the work that leads to your answer.

- Find $\lim_{x \rightarrow 0} \frac{f(x) - \frac{1}{3}}{x}$.

- Write the first three nonzero terms and the general term for an infinite series that represents $\int_0^1 f(x) dx$.
- Find the sum of the series determined in part (c).

$$(a) e^u = 1 + u + \frac{u^2}{2} + \frac{u^3}{3!} + \dots + \frac{u^n}{n!} + \dots$$

$$e^{-2x^2} = 1 - 2x^2 + \frac{4x^4}{2} - \frac{8x^6}{3!} + \dots + \frac{(-1)^n 2^n x^{2n}}{n!} + \dots$$

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(b) The series for e^u converges for $-\infty < u < \infty$

So the series for e^{-2x^2} converges for $-\infty < -2x^2 < \infty$

And, thus, for $-\infty < x < \infty$

Or

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{(-1)^{n+1} 2^{n+1} x^{2(n+1)}}{(n+1)!} \cdot \frac{n!}{(-1)^n 2^n x^{2n}} \right| \\ &= \lim_{n \rightarrow \infty} \frac{2}{n+1} x^2 = 0 < 1 \end{aligned}$$

So the series for e^{-2x^2} converges for $-\infty < x < \infty$

$$(c) f(x) - g(x) = \frac{16x^8}{4!} - \frac{32x^{16}}{5!} + \dots$$

This is an alternative series for each x , since the powers of x are even.

Also, $\left| \frac{a_{n+1}}{a_n} \right| = \frac{2}{n+1} x^2 < 1$ for $-0.6 \leq x \leq 0.6$ so terms are decreasing in absolute value.

$$\begin{aligned} \text{Thus } |f(x) - g(x)| &\leq \frac{16x^8}{4!} \leq \frac{16(0.6)^8}{4!} \\ &= 0.011\dots < 0.02 \end{aligned}$$

1997 BC2

Solution

(a) $f(4) = P(4) = 7$

$$\frac{f'''(4)}{3!} = -2, \quad f'''(4) = -12$$

(b) $P_3(x) = 7 - 3(x-4) + 5(x-4)^2 - 2(x-4)^3$

$$P_3'(x) = -3 + 10(x-4) - 6(x-4)^2$$

$$f'(4.3) \approx -3 + 10(0.3) - 6(0.3)^2 = -0.54$$

(c) $P_4(g, x) = \int_4^x P_3(t) dt$

$$= \int_4^x [7 - 3(t-4) + 5(t-4)^2 - 2(t-4)^3] dt$$

$$= 7(x-4) - \frac{3}{2}(x-4)^2 + \frac{5}{3}(x-4)^3 - \frac{1}{2}(x-4)^4$$

(d) No. The information given provides values for $f(4), f'(4), f''(4), f'''(4)$ and $f^{(4)}(4)$ only.

#2

1996 BC2

Solution

(a) $a_n = \frac{f^{(n)}(0)}{n!} = \frac{1}{(n+1)!}$

$$f'(0) = a_1 = \frac{1}{2}$$

$$f^{(17)}(0) = 17! a_{17} = 17! \left(\frac{1}{18!} \right) = \frac{1}{18}$$

(b)

$$\lim_{n \rightarrow \infty} \frac{\frac{|x^{n+1}|}{(n+2)!}}{\frac{|x^n|}{(n+1)!}} = \lim_{n \rightarrow \infty} \frac{|x|}{n+2} = 0 < 1$$

Converges for all x , by ratio test

(c) $g(x) = xf(x)$

$$= x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^{n+1}}{(n+1)!} + \cdots$$

(d) $e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots$

$$e^x - 1 = g(x) = xf(x)$$

$$f(x) = \begin{cases} \frac{e^x - 1}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$$

#3

$$(a) T_2(x) = 3 + (-2)(x-1) + \frac{2}{2}(x-1)^2$$

$$f(0.7) \approx 3 + 0.6 + 0.09 = 3.69$$

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$$(b) T_3(x) = 3 - 2(x-1) + (x-1)^2 + \frac{4}{6}(x-1)^3$$

$$f(1.2) \approx 3 - 0.4 + 0.04 + \frac{2}{3}(0.008) = 2.645$$

$$(c) T_3'(x) = -2 + 2(x-1) + 2(x-1)^2$$

$$f'(1.2) \approx -2 + 0.4 + 0.08 = -1.52$$

(a) Taylor approach

Geometric Approach

$$f(2) = 1$$

$$f'(2) = -(2-1)^{-2} = -1$$

$$f''(2) = 2(2-1)^{-3} = 2; \quad \frac{f''(2)}{2!} = 1$$

$$f'''(2) = -6(2-1)^{-4} = -6; \quad \frac{f'''(2)}{3!} = -1$$

$$\frac{1}{x-1} = \frac{1}{1+(x-2)}$$

$$= 1 - u + u^2 - u^3 + \dots + (-1)^n u^n + \dots$$

where $u = x - 2$

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Therefore $\frac{1}{x-1} = 1 - (x-2) + (x-2)^2 - (x-2)^3 + \dots + (-1)^n (x-2)^n + \dots$

(b) Antidifferentiates series in (a):

$$\ln|x-1| = C + x - \frac{1}{2}(x-2)^2 + \frac{1}{3}(x-2)^3 - \frac{1}{4}(x-2)^4 + \dots + \frac{(-1)^n (x-2)^{n+1}}{n+1} + \dots$$

$$0 = \ln|2-1| \Rightarrow C = -2$$

Note: If $C \neq 0$, "first 4 terms" need not include $-\frac{1}{4}(x-2)^4$

$$(c) \quad \ln \frac{3}{2} = \ln \left| \frac{5}{2} - 1 \right| = \frac{1}{2} - \frac{1}{2} \left(\frac{1}{2} \right)^2 + \frac{1}{3} \left(\frac{1}{2} \right)^3 - \dots$$

$$= \frac{1}{2} - \frac{1}{8} + \frac{1}{24} - \dots$$

since $\frac{1}{24} < \frac{1}{20}$, $\frac{1}{2} - \frac{1}{8} = 0.375$ is sufficient.

Justification: Since series is alternating, with terms convergent to 0 and decreasing in absolute value, the truncation error is less than the first omitted term.

Alternate Justification: $|R_n| = \left| \frac{1}{(C-1)^{n+1}} \frac{1}{n+1} \left(\frac{1}{2} \right)^{n+1} \right|$, where $2 < C < \frac{5}{2}$

$$< \frac{1}{n+1} \frac{1}{2^{n+1}}$$

$$< \frac{1}{20} \text{ when } n \geq 2$$

$$(a) \ln\left(\frac{1}{1+3x}\right) = \ln\left(\frac{1}{1-(-3x)}\right)$$

$$= \sum_{n=1}^{\infty} \frac{(-3x)^n}{n} \text{ or } \sum_{n=1}^{\infty} (-1)^n \frac{3^n}{n} x^n$$

We must have $-1 \leq -3x < 1$, so interval of convergence is $-\frac{1}{3} < x \leq \frac{1}{3}$.

$$(b) \sum_{n=1}^{\infty} \frac{(-1)^n}{n} = \ln\left(\frac{1}{1-(-1)}\right) = \ln\left(\frac{1}{2}\right)$$

(c) Some p such that $0 < p \leq \frac{1}{2}$ because $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^p}$ converges by AST, but the p -series $\sum_{n=1}^{\infty} \frac{1}{n^{2p}}$ diverges for $2p \leq 1$.

(d) Some p such that $\frac{1}{2} < p \leq 1$ because the p -series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges for $p \leq 1$ and the p -series $\sum_{n=1}^{\infty} \frac{1}{n^{2p}}$ converges for $2p > 1$.

2 $\left\{ \begin{array}{l} 1 : \text{series} \\ 1 : \text{interval of convergence} \end{array} \right.$

#6

1 : answer

3 $\left\{ \begin{array}{l} 1 : \text{correct } p \\ 1 : \text{reason why } \sum \frac{(-1)^n}{n^p} \text{ converges} \\ 1 : \text{reason why } \sum \frac{1}{n^{2p}} \text{ diverges} \end{array} \right.$

3 $\left\{ \begin{array}{l} 1 : \text{correct } p \\ 1 : \text{reason why } \sum \frac{1}{n^p} \text{ diverges} \\ 1 : \text{reason why } \sum \frac{1}{n^{2p}} \text{ converges} \end{array} \right.$

$$(a) T_3(f, 2)(x) = -3 + 5(x - 2) + \frac{3}{2}(x - 2)^2 - \frac{8}{6}(x - 2)^3$$

$$f(1.5) \approx T_3(f, 2)(1.5)$$

$$= -3 + 5(-0.5) + \frac{3}{2}(-0.5)^2 - \frac{4}{3}(-0.5)^3$$

$$= -4.958\bar{3} = -4.958$$

$$(b) \text{Lagrange Error Bound} = \frac{3}{4!}|1.5 - 2|^4 = 0.0078125$$

$$f(1.5) > -4.958\bar{3} - 0.0078125 = -4.966 > -5$$

Therefore, $f(1.5) \neq -5$.

$$(c) P(x) = T_4(g, 0)(x)$$

$$= T_2(f, 2)(x^2 + 2) = -3 + 5x^2 + \frac{3}{2}x^4$$

The coefficient of x in $P(x)$ is $g'(0)$. This coefficient is 0, so $g'(0) = 0$.

The coefficient of x^2 in $P(x)$ is $\frac{g''(0)}{2!}$. This coefficient is 5, so $g''(0) = 10$ which is greater than 0.

Therefore, g has a relative minimum at $x = 0$.

$$4 \left\{ \begin{array}{l} 3: T_3(f, 2)(x) \\ <-1> \text{ each error} \\ 1: \text{ approximation of } f(1.5) \end{array} \right.$$

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$$2 \left\{ \begin{array}{l} 1: \text{ value of Lagrange Error Bound} \\ 1: \text{ explanation} \end{array} \right.$$

$$3 \left\{ \begin{array}{l} 2: T_4(g, 0)(x) \\ <-1> \text{ each incorrect, missing,} \\ \quad \text{or extra term} \\ 1: \text{ explanation} \end{array} \right.$$

Note:

<-1> max for improper use of + ... or equality

(a) $P_6(x) = 7 + \frac{1!}{3^2} \cdot \frac{1}{2!} (x-2)^2 + \frac{3!}{3^4} \cdot \frac{1}{4!} (x-2)^4 + \frac{5!}{3^6} \cdot \frac{1}{6!} (x-2)^6$

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(b) $\frac{(2n-1)!}{3^{2n}} \cdot \frac{1}{(2n)!} = \frac{1}{3^{2n}(2n)}$

(c) The Taylor series for f about $x = 2$ is

$$f(x) = 7 + \sum_{n=1}^{\infty} \frac{1}{2n \cdot 3^{2n}} (x-2)^{2n}.$$

$$L = \lim_{n \rightarrow \infty} \left| \frac{\frac{1}{2(n+1)} \cdot \frac{1}{3^{2(n+1)}} (x-2)^{2(n+1)}}{\frac{1}{2n} \cdot \frac{1}{3^{2n}} (x-2)^{2n}} \right|$$
$$= \lim_{n \rightarrow \infty} \left| \frac{2n}{2(n+1)} \cdot \frac{3^{2n}}{3^2 3^{2n}} (x-2)^2 \right| = \frac{(x-2)^2}{9}$$

$L < 1$ when $|x-2| < 3$.

Thus, the series converges when $-1 < x < 5$.

When $x = 5$, the series is $7 + \sum_{n=1}^{\infty} \frac{3^{2n}}{2n \cdot 3^{2n}} = 7 + \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n}$,

which diverges, because $\sum_{n=1}^{\infty} \frac{1}{n}$, the harmonic series, diverges.

When $x = -1$, the series is $7 + \sum_{n=1}^{\infty} \frac{(-3)^{2n}}{2n \cdot 3^{2n}} = 7 + \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n}$,

which diverges, because $\sum_{n=1}^{\infty} \frac{1}{n}$, the harmonic series, diverges.

The interval of convergence is $(-1, 5)$.

(a) $f(2) = 1; f'(2) = \frac{2!}{3}; f''(2) = \frac{3!}{3^2}; f'''(2) = \frac{4!}{3^3}$
 $f(x) = 1 + \frac{2}{3}(x-2) + \frac{3!}{2!3^2}(x-2)^2 + \frac{4!}{3!3^3}(x-2)^3 +$
 $+ \dots + \frac{(n+1)!}{n!3^n}(x-2)^n + \dots$
 $= 1 + \frac{2}{3}(x-2) + \frac{3}{3^2}(x-2)^2 + \frac{4}{3^3}(x-2)^3 +$
 $+ \dots + \frac{n+1}{3^n}(x-2)^n + \dots$

(b) $\lim_{n \rightarrow \infty} \left| \frac{\frac{n+2}{3^{n+1}}(x-2)^{n+1}}{\frac{n+1}{3^n}(x-2)^n} \right| = \lim_{n \rightarrow \infty} \frac{n+2}{n+1} \cdot \frac{1}{3} |x-2|$
 $= \frac{1}{3} |x-2| < 1$ when $|x-2| < 3$

The radius of convergence is 3.

(c) $g(2) = 3; g'(2) = f(2); g''(2) = f'(2); g'''(2) = f''(2)$
 $g(x) = 3 + (x-2) + \frac{1}{3}(x-2)^2 + \frac{1}{3^2}(x-2)^3 +$
 $+ \dots + \frac{1}{3^n}(x-2)^{n+1} + \dots$

(d) No, the Taylor series does not converge at $x = -2$ because the geometric series only converges on the interval $|x-2| < 3$.

3 : $\left\{ \begin{array}{l} 1 : \text{coefficients } \frac{f^{(n)}(2)}{n!} \text{ in} \\ \text{first four terms} \\ 1 : \text{powers of } (x-2) \text{ in} \\ \text{first four terms} \\ 1 : \text{general term} \end{array} \right.$

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3 : $\left\{ \begin{array}{l} 1 : \text{sets up ratio} \\ 1 : \text{limit} \\ 1 : \text{applies ratio test to} \\ \text{conclude radius of} \\ \text{convergence is 3} \end{array} \right.$

2 : $\left\{ \begin{array}{l} 1 : \text{first four terms} \\ 1 : \text{general term} \end{array} \right.$

1 : answer with reason

$$(a) \lim_{n \rightarrow \infty} \left| \frac{(n+2)x^{n+1}}{3^{n+2}} \cdot \frac{3^{n+1}}{(n+1)x^n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(n+2)x}{(n+1)3} \right| = \left| \frac{x}{3} \right| < 1$$

At $x = -3$, the series is $\sum_{n=0}^{\infty} (-1)^n \frac{n+1}{3}$, which diverges.

At $x = 3$, the series is $\sum_{n=0}^{\infty} \frac{n+1}{3}$, which diverges.

Therefore, the interval of convergence is $-3 < x < 3$.

$$(b) \lim_{x \rightarrow 0} \frac{f(x) - \frac{1}{3}}{x} = \lim_{x \rightarrow 0} \left(\frac{2}{3^2} + \frac{3}{3^3}x + \frac{4}{3^4}x^2 + \dots \right) = \frac{2}{9}$$

$$(c) \int_0^1 f(x) dx = \int_0^1 \left(\frac{1}{3} + \frac{2}{3^2}x + \frac{3}{3^3}x^2 + \dots + \frac{n+1}{3^{n+1}}x^n + \dots \right) dx$$

$$= \left(\frac{1}{3}x + \frac{1}{3^2}x^2 + \frac{1}{3^3}x^3 + \dots + \frac{1}{3^{n+1}}x^{n+1} + \dots \right) \Big|_{x=0}^{x=1}$$

$$= \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \dots + \frac{1}{3^{n+1}} + \dots$$

(d) The series representing $\int_0^1 f(x) dx$ is a geometric series.

$$\text{Therefore, } \int_0^1 f(x) dx = \frac{\frac{1}{3}}{1 - \frac{1}{3}} = \frac{1}{2}.$$

1 : sets up ratio test
1 : computes limit
4 : 1 : conclusion of ratio test
1 : endpoint conclusion

#10

1 : answer

1 : antidifferentiation
of series
3 : 1 : first three terms for
definite integral series
1 : general term

1 : answer