Free Response Practice for Series

1. Let f be the function given by $f(x) = e^{-2x^2}$.

(a) Find the first four nonzero terms and the general term of the power series for $f(x)$ about $x = 0$.

(b) Find the interval of convergence of the power series for $f(x)$ about $x = 0$. Show the analysis that leads to your conclusion.

(c) Let g be the function given by the sum of the first four nonzero terms of the power series for $f(x)$ about $x = 0$. Show that $|f(x) - g(x)| < 0.02$ for $-0.6 \le x \le 0.6$.

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2. Let $P(x) = 7 - 3(x-4) + 5(x-4)^2 - 2(x-4)^3 + 6(x-4)^4$ be the fourth-degree Taylor polynomial for the function f about 4. Assume f has derivatives of all orders for all real numbers.

(a) Find $f(4)$ and $f'''(4)$.

(b) Write the second-degree Taylor polynomial for f' about 4 and use it to approximate $f'(4.3)$.

(c) Write the fourth-degree Taylor polynomial for $g(x) = \int_4^x f(t)\,dt$ about 4.

(d) Can $f(3)$ be determined from the information given? Justify your answer.

3. The Maclaurin series for $f(x)$ is given by $1+\frac{x}{2!}+\frac{x^2}{3!}+\frac{x^3}{4!}+\cdots+\frac{x^n}{(n+1)!}+\cdots$

(a) Find $f'(0)$ and $f^{(17)}(0)$.

(b) For what values of x does the given series converge? Show your reasoning.

(c) Let $g(x)=x\,f(x)$. Write the Maclaurin series for $g(x)$, showing the first three nonzero terms and the general term.

(d) Write $g(x)$ in terms of a familiar function without using series. Then, write $f(x)$ in terms of the same familiar function.

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4. Let f be a function that has derivatives of all orders for all real numbers. Assume $f(1)=3$, $f'(1)=-2$, $f''(1)=2$, and $f'''(1)=4$.

(a) Write the second-degree Taylor polynomial for f about $x=1$ and use it to approximate $f(0.7)$.

(b) Write the third-degree Taylor polynomial for f about $x=1$ and use it to approximate $f(1.2)$.

(c) Write the second-degree Taylor polynomial for f', the derivative of f, about $x=1$ and use it to approximate $f'(1.2)$.

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5. Let f be the function defined by $f(x)=\frac{1}{x-1}$.

(a) Write the first four terms and the general term of the Taylor series expansion of $f(x)$ about $x=2$.

(b) Use the result from part (a) to find the first four terms and the general term of the series expansion about $x=2$ for $\ln|x-1|$.

(c) Use the series in part (b) to compute a number that differs from $\ln\frac{3}{2}$ by less than 0.05. Justify your answer.

The Maclaurin series for $\ln\left(\frac{1}{1-x}\right)$ is $\sum_{n=1}^{\infty} \frac{x^n}{n}$ with interval of convergence $-1 \le x < 1$.

6. (a) Find the Maclaurin series for $\ln\left(\frac{1}{1+3x}\right)$ and determine the interval of convergence.

(b) Omit

(c) Give a value of p such that $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^p}$ converges, but $\sum_{n=1}^{\infty} \frac{1}{n^{2p}}$ diverges. Give reasons why your value of p is correct.

(d) Give a value of p such that $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges, but $\sum_{n=1}^{\infty} \frac{1}{n^{2p}}$ converges. Give reasons why your value of p is correct.

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7. The function f has derivatives of all orders for all real numbers x. Assume $f(2) = -3$, $f'(2) = 5$, $f''(2) = 3$, and $f'''(2) = -8$.

(a) Write the third-degree Taylor polynomial for f about $x = 2$ and use it to approximate $f(1.5)$.

(b) The fourth derivative of f satisfies the inequality $|f^{(4)}(x)| \le 3$ for all x in the closed interval $[1.5, 2]$. Use the Lagrange error bound on the approximation to $f(1.5)$ found in part (a) to explain why $f(1.5) \ne -5$.

(c) Write the fourth-degree Taylor polynomial, $P(x)$, for $g(x) = f(x^2 + 2)$ about $x = 0$. Use P to explain why g must have a relative minimum at $x = 0$.

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8. Let f be a function with derivatives of all orders and for which $f(2) = 7$. When n is odd, the nth derivative of f at $x = 2$ is 0. When n is even and $n \ge 2$, the nth derivative of f at $x = 2$ is given by $f^{(n)}(2) = \frac{(n-1)!}{3^n}$.

(a) Write the sixth-degree Taylor polynomial for f about $x = 2$.

(b) In the Taylor series for f about $x = 2$, what is the coefficient of $(x-2)^{2n}$ for $n \ge 1$?

(c) Find the interval of convergence of the Taylor series for f about $x = 2$. Show the work that leads to your answer.

9. The function f has a Taylor series about $x = 2$ that converges to $f(x)$ for all x in the interval of convergence. The nth derivative of f at $x = 2$ is given by $f^{(n)}(2) = \dfrac{(n+1)!}{3^n}$ for $n \geq 1$, and $f(2) = 1$.

(a) Write the first four terms and the general term of the Taylor series for f about $x = 2$.

(b) Find the radius of convergence for the Taylor series for f about $x = 2$. Show the work that leads to your answer.

(c) Let g be a function satisfying $g(2) = 3$ and $g'(x) = f(x)$ for all x. Write the first four terms and the general term of the Taylor series for g about $x = 2$.

(d) Does the Taylor series for g as defined in part (c) converge at $x = -2$? Give a reason for your answer.

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10 A function f is defined by

$$f(x) = \frac{1}{3} + \frac{2}{3^2}x + \frac{3}{3^3}x^2 + \cdots + \frac{n+1}{3^{n+1}}x^n + \cdots$$

for all x in the interval of convergence of the given power series.

(a) Find the interval of convergence for this power series. Show the work that leads to your answer.

(b) Find $\displaystyle\lim_{x \to 0} \frac{f(x) - \frac{1}{3}}{x}$.

(c) Write the first three nonzero terms and the general term for an infinite series that represents $\displaystyle\int_0^1 f(x)\,dx$.

(d) Find the sum of the series determined in part (c).

1994 BC 5

(a) $e^u = 1 + u + \frac{u^2}{2} + \frac{u^3}{3!} + \cdots + \frac{u^n}{n!} + \cdots$

$e^{-2x^2} = 1 - 2x^2 + \frac{4x^4}{2} - \frac{8x^6}{3!} + \cdots + \frac{(-1)^n 2^n x^{2n}}{n!} + \cdots$

#1

(b) The series for e^u converges for $-\infty < u < \infty$

So the series for e^{-2x^2} converges for $-\infty < -2x^2 < \infty$

And, thus, for $-\infty < x < \infty$

Or

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1} 2^{n+1} x^{2(n+1)}}{(n+1)!}\cdot\frac{n!}{(-1)^n 2^n x^{2n}}\right|$$
$$= \lim_{n\to\infty}\frac{2}{n+1}x^2 = 0 < 1$$

So the series for e^{-2x^2} converges for $-\infty < x < \infty$

(c) $f(x) - g(x) = \frac{16x^8}{4!} - \frac{32x^{10}}{5!} + \cdots$

This is an alternative series for each x, since the powers of x are even.

Also, $\left|\frac{a_n + 1}{a_n}\right| = \frac{2}{n+1}x^2 < 1$ for $-0.6 \le x \le 0.6$ so terms are decreasing in absolute value.

Thus $|f(x) - g(x)| \le \frac{16x^8}{4!} \le \frac{16(0.6)^8}{4!}$
$= 0.011\cdots < 0.02$

1997 BC2
Solution

#2

(a) $f(4)=P(4)=7$

$\dfrac{f'''(4)}{3!}=-2,\ f'''(4)=-12$

(b) $P_3(x)=7-3(x-4)+5(x-4)^2-2(x-4)^3$

$P_3'(x)=-3+10(x-4)-6(x-4)^2$

$f'(4.3)\approx -3+10(0.3)-6(0.3)^2=-0.54$

(c) $P_4(g,x)=\int_4^x P_3(t)\,dt$

$=\int_4^x\left[7-3(t-4)+5(t-4)^2-2(t-4)^3\right]dt$

$=7(x-4)-\dfrac{3}{2}(x-4)^2+\dfrac{5}{3}(x-4)^3-\dfrac{1}{2}(x-4)^4$

(d) No. The information given provides values for $f(4), f'(4), f''(4), f'''(4)$ and $f^{(4)}(4)$ only.

1996 BC2
Solution

#3

(a) $a_n=\dfrac{f^{(n)}(0)}{n!}=\dfrac{1}{(n+1)!}$

$f'(0)=a_1=\dfrac{1}{2}$

$f^{(17)}(0)=17!a_{17}=17!\left(\dfrac{1}{18!}\right)=\dfrac{1}{18}$

(b)

$$\lim_{n\to\infty}\frac{\left|\dfrac{x^{n+1}}{(n+2)!}\right|}{\left|\dfrac{x^n}{(n+1)!}\right|}=\lim_{n\to\infty}\frac{|x|}{n+2}=0<1$$

Converges for all x, by ratio test

(c) $g(x)=xf(x)$

$=x+\dfrac{x^2}{2!}+\dfrac{x^3}{3!}+\cdots+\dfrac{x^{n+1}}{(n+1)!}+\cdots$

(d) $e^x=1+x+\dfrac{x^2}{2!}+\cdots+\dfrac{x^n}{n!}+\cdots$

$e^x-1=g(x)=xf(x)$

$$f(x)=\begin{cases}\dfrac{e^x-1}{x} & \text{if } x\neq 0\\ 1 & \text{if } x=0\end{cases}$$

#4

(a) $T_2(x) = 3 + (-2)(x-1) + \frac{2}{2}(x-1)^2$

$f(0.7) \approx 3 + 0.6 + 0.09 = 3.69$

(b) $T_3(x) = 3 - 2(x-1) + (x-1)^2 + \frac{4}{6}(x-1)^3$

$f(1.2) \approx 3 - 0.4 + 0.04 + \frac{2}{3}(0.008) = 2.645$

(c) $T_3'(x) = -2 + 2(x-1) + 2(x-1)^2$

$f'(1.2) \approx -2 + 0.4 + 0.08 = -1.52$

#5

(a) Taylor approach

$f(2) = 1$

$f'(2) = -(2-1)^{-2} = -1$

$f''(2) = 2(2-1)^{-3} = 2; \quad \frac{f''(2)}{2!} = 1$

$f'''(2) = -6(2-1)^{-4} = -6; \frac{f'''(2)}{3!} = -1$

Geometric Approach

$$\frac{1}{x-1} = \frac{1}{1+(x-2)}$$

$$= 1 - u + u^2 - u^3 + \cdots + (-1)^n u^n + \cdots$$

where $u = x - 2$

Therefore $\frac{1}{x-1} = 1 - (x-2) + (x-2)^2 - (x-2)^3 + \cdots + (-1)^n (x-2)^n + \cdots$

(b) Antidifferentiates series in (a):

$$\ln|x-1| = C + x - \frac{1}{2}(x-2)^2 + \frac{1}{3}(x-2)^3 - \frac{1}{4}(x-2)^4 + \cdots + \frac{(-1)^n (x-2)^{n+1}}{n+1} + \cdots$$

$0 = \ln|2-1| \Rightarrow C = -2$

Note: If $C \neq 0$, "first 4 terms" need not include $-\frac{1}{4}(x-2)^4$

(c) $\ln\frac{3}{2} = \ln\left|\frac{5}{2} - 1\right| = \frac{1}{2} - \frac{1}{2}\left(\frac{1}{2}\right)^2 + \frac{1}{3}\left(\frac{1}{2}\right)^3 - \cdots$

$= \frac{1}{2} - \frac{1}{8} + \frac{1}{24} - \cdots$

since $\frac{1}{24} < \frac{1}{20}$, $\frac{1}{2} - \frac{1}{8} = 0.375$ is sufficient.

Justification: Since series is alternating, with terms convergent to 0 and decreasing in absolute value, the truncation error is less than the first omitted term.

Alternate Justification: $|R_n| = \left|\frac{1}{(C-1)^{n+1}} \frac{1}{n+1}\left(\frac{1}{2}\right)^{n+1}\right|$, where $2 < C < \frac{5}{2}$

$< \frac{1}{n+1}\frac{1}{2^{n+1}}$

$< \frac{1}{20}$ when $n \geq 2$

#6

(a) $\ln\left(\frac{1}{1+3x}\right) = \ln\left(\frac{1}{1-(-3x)}\right)$

$$= \sum_{n=1}^{\infty} \frac{(-3x)^n}{n} \text{ or } \sum_{n=1}^{\infty} (-1)^n \frac{3^n}{n} x^n$$

We must have $-1 \le -3x < 1$, so interval of convergence is $-\frac{1}{3} < x \le \frac{1}{3}$.

2 { 1 : series; 1 : interval of convergence }

(b) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n} = \ln\left(\frac{1}{1-(-1)}\right) = \ln\left(\frac{1}{2}\right)$

1 : answer

(c) Some p such that $0 < p \le \frac{1}{2}$ because $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^p}$ converges by AST, but the p-series $\sum_{n=1}^{\infty} \frac{1}{n^{2p}}$ diverges for $2p \le 1$.

3 { 1 : correct p; 1 : reason why $\sum \frac{(-1)^n}{n^p}$ converges; 1 : reason why $\sum \frac{1}{n^{2p}}$ diverges }

(d) Some p such that $\frac{1}{2} < p \le 1$ because the p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges for $p \le 1$ and the p-series $\sum_{n=1}^{\infty} \frac{1}{n^{2p}}$ converges for $2p > 1$.

3 { 1 : correct p; 1 : reason why $\sum \frac{1}{n^p}$ diverges; 1 : reason why $\sum \frac{1}{n^{2p}}$ converges }

#7

(a) $T_3(f,2)(x) = -3 + 5(x-2) + \frac{3}{2}(x-2)^2 - \frac{8}{6}(x-2)^3$

$f(1.5) \approx T_3(f,2)(1.5)$

$= -3 + 5(-0.5) + \frac{3}{2}(-0.5)^2 - \frac{4}{3}(-0.5)^3$

$= -4.958\overline{3} = -4.958$

4 { 3: $T_3(f,2)(x)$
< −1 > each error
1: approximation of $f(1.5)$ }

(b) Lagrange Error Bound $= \frac{3}{4!}|1.5 - 2|^4 = 0.0078125$

$f(1.5) > -4.958\overline{3} - 0.0078125 = -4.966 > -5$

Therefore, $f(1.5) \neq -5$.

2 { 1: value of Lagrange Error Bound
1: explanation }

(c) $P(x) = T_4(g,0)(x)$

$= T_2(f,2)(x^2+2) = -3 + 5x^2 + \frac{3}{2}x^4$

The coefficient of x in $P(x)$ is $g'(0)$. This coefficient is 0, so $g'(0) = 0$.

The coefficient of x^2 in $P(x)$ is $\frac{g''(0)}{2!}$. This coefficient is 5, so $g''(0) = 10$ which is greater than 0.

Therefore, g has a relative minimum at $x = 0$.

3 { 2: $T_4(g,0)(x)$
< −1 > each incorrect, missing, or extra term
1: explanation }

Note:
< −1 > max for improper use of + . . . or equality

#8

(a) $P_6(x) = 7 + \frac{1!}{3^2}\cdot\frac{1}{2!}(x-2)^2 + \frac{3!}{3^4}\cdot\frac{1}{4!}(x-2)^4 + \frac{5!}{3^6}\cdot\frac{1}{6!}(x-2)^6$

(b) $\frac{(2n-1)!}{3^{2n}}\cdot\frac{1}{(2n)!} = \frac{1}{3^{2n}(2n)}$

(c) The Taylor series for f about $x = 2$ is

$$f(x) = 7 + \sum_{n=1}^{\infty}\frac{1}{2n\cdot 3^{2n}}(x-2)^{2n}.$$

$$L = \lim_{n\to\infty}\left|\frac{\frac{1}{2(n+1)}\cdot\frac{1}{3^{2(n+1)}}(x-2)^{2(n+1)}}{\frac{1}{2n}\cdot\frac{1}{3^{2n}}(x-2)^{2n}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{2n}{2(n+1)}\cdot\frac{3^{2n}}{3^2 3^{2n}}(x-2)^2\right| = \frac{(x-2)^2}{9}$$

$L < 1$ when $|x-2| < 3$.

Thus, the series converges when $-1 < x < 5$.

When $x = 5$, the series is $7 + \sum_{n=1}^{\infty}\frac{3^{2n}}{2n\cdot 3^{2n}} = 7 + \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n}$,

which diverges, because $\sum_{n=1}^{\infty}\frac{1}{n}$, the harmonic series, diverges.

When $x = -1$, the series is $7 + \sum_{n=1}^{\infty}\frac{(-3)^{2n}}{2n\cdot 3^{2n}} = 7 + \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n}$,

which diverges, because $\sum_{n=1}^{\infty}\frac{1}{n}$, the harmonic series, diverges.

The interval of convergence is $(-1, 5)$.

#9

(a) $f(2) = 1;\ f'(2) = \frac{2!}{3};\ f''(2) = \frac{3!}{3^2};\ f'''(2) = \frac{4!}{3^3}$

$$f(x) = 1 + \frac{2}{3}(x-2) + \frac{3!}{2!3^2}(x-2)^2 + \frac{4!}{3!3^3}(x-2)^3 + \cdots + \frac{(n+1)!}{n!3^n}(x-2)^n + \cdots$$

$$= 1 + \frac{2}{3}(x-2) + \frac{3}{3^2}(x-2)^2 + \frac{4}{3^3}(x-2)^3 + \cdots + \frac{n+1}{3^n}(x-2)^n + \cdots$$

3 : { 1 : coefficients $\frac{f^{(n)}(2)}{n!}$ in first four terms; 1 : powers of $(x-2)$ in first four terms; 1 : general term }

(b) $$\lim_{n\to\infty}\left|\frac{\frac{n+2}{3^{n+1}}(x-2)^{n+1}}{\frac{n+1}{3^n}(x-2)^n}\right| = \lim_{n\to\infty}\frac{n+2}{n+1}\cdot\frac{1}{3}|x-2|$$

$= \frac{1}{3}|x-2| < 1$ when $|x-2| < 3$

The radius of convergence is 3.

3 : { 1 : sets up ratio; 1 : limit; 1 : applies ratio test to conclude radius of convergence is 3 }

(c) $g(2) = 3;\ g'(2) = f(2);\ g''(2) = f'(2);\ g'''(2) = f''(2)$

$$g(x) = 3 + (x-2) + \frac{1}{3}(x-2)^2 + \frac{1}{3^2}(x-2)^3 + \cdots + \frac{1}{3^n}(x-2)^{n+1} + \cdots$$

2 : { 1 : first four terms; 1 : general term }

(d) No, the Taylor series does not converge at $x = -2$ because the geometric series only converges on the interval $|x-2| < 3$.

1 : answer with reason

(a) $\lim_{n\to\infty}\left|\dfrac{\dfrac{(n+2)x^{n+1}}{3^{n+2}}}{\dfrac{(n+1)x^n}{3^{n+1}}}\right| = \lim_{n\to\infty}\left|\dfrac{(n+2)}{(n+1)}\dfrac{x}{3}\right| = \left|\dfrac{x}{3}\right| < 1$

At $x=-3$, the series is $\sum_{n=0}^{\infty}(-1)^n\dfrac{n+1}{3}$, which diverges.

At $x=3$, the series is $\sum_{n=0}^{\infty}\dfrac{n+1}{3}$, which diverges.

Therefore, the interval of convergence is $-3 < x < 3$.

4 :
- 1 : sets up ratio test
- 1 : computes limit
- 1 : conclusion of ratio test
- 1 : endpoint conclusion

#10

(b) $\lim_{x\to 0}\dfrac{f(x)-\frac{1}{3}}{x} = \lim_{x\to 0}\left(\dfrac{2}{3^2}+\dfrac{3}{3^3}x+\dfrac{4}{3^4}x^2+\cdots\right) = \dfrac{2}{9}$

1 : answer

(c) $\int_0^1 f(x)\,dx = \int_0^1\left(\dfrac{1}{3}+\dfrac{2}{3^2}x+\dfrac{3}{3^3}x^2+\cdots+\dfrac{n+1}{3^{n+1}}x^n+\cdots\right)dx$

$= \left(\dfrac{1}{3}x+\dfrac{1}{3^2}x^2+\dfrac{1}{3^3}x^3+\cdots+\dfrac{1}{3^{n+1}}x^{n+1}+\cdots\right)\Big|_{x=0}^{x=1}$

$= \dfrac{1}{3}+\dfrac{1}{3^2}+\dfrac{1}{3^3}+\cdots+\dfrac{1}{3^{n+1}}+\cdots$

3 :
- 1 : antidifferentiation of series
- 1 : first three terms for definite integral series
- 1 : general term

(d) The series representing $\int_0^1 f(x)\,dx$ is a geometric series.

Therefore, $\int_0^1 f(x)\,dx = \dfrac{\frac{1}{3}}{1-\frac{1}{3}} = \dfrac{1}{2}$.

1 : answer