Euler’s Method and Logistic Growth (BC Only)

Euler’s Method
Students should be able to:
- Approximate numerical solutions of differential equations using Euler’s method without a calculator.
- Recognize the method as a recursion formula extension of the point-slope version of the equation of a tangent line.
- General Format: \( y - y_1 = m(x - x_1) \) (point-slope form of equation of tangent line)
  \[ y = y_1 + m(x - x_1) \] (where \( \frac{dy}{dx} \) is given differential equation, \( x - x_1 \) is the step size (\( h \) or \( \Delta x \), and \( y_1 \) is the given \( y \) the first time, for other iterations it is the previous \( y \) or \( y_{n-1} \))
- Produce ordered pairs of points for the function to be approximated given an increment for the change in \( x \).

Logistic Growth
Students should be able to:
- Know that the solution of the general logistic differential equation is
  \[ \frac{dP}{dt} = kP(L - P) \quad \text{is} \quad P = \frac{L}{1 + ae^{-kt}} \]
  where \( L \) is the maximum carrying capacity and \( k \) is the growth constant (\( L \) and \( k \) are both positive).
- Determine the limit of the population over a long period of time (always the maximum carrying capacity \( L \), regardless of the initial population)
- Determine when the population is growing the fastest (always when the population is half of the maximum carrying capacity)
- Factor the differential equation (if not given in factored form) so that it is in the form of
  \[ \frac{dP}{dt} = kP(L - P) \]
  where \( L \), the maximum carrying capacity, can be easily determined.
  \( L \) is the maximum population that can be sustained or supported as time \( t \) increases. A population that satisfies the differential equation does not grow without bound (as in exponential growth) but approaches the carrying capacity \( L \) as time \( t \) increases.
Multiple Choice Euler’s Method

1. (calculator not allowed)

   Let \( y = f(x) \) be the solution to the differential equation \( \frac{dy}{dx} = x + y \) with the initial condition \( f(1) = 2 \). What is the approximation for \( f(2) \) if Euler’s method is used, starting at \( x = 1 \) with a step size of 0.5?

   (A) 3
   (B) 5
   (C) 6
   (D) 10
   (E) 12

2. (calculator not allowed)

   Given that \( y(1) = -3 \) and \( \frac{dy}{dx} = 2x + y \), what is the approximation for \( y(2) \) if Euler’s method is used with a step size of 0.5, starting at \( x = 1 \)?

   (A) -5
   (B) -4.25
   (C) -4
   (D) -3.75
   (E) -3.5

3. (calculator not allowed)

   Let \( y = f(x) \) be the particular solution to the differential equation \( \frac{dy}{dx} = x + 2y \) with the initial condition \( f(0) = 1 \). Use Euler’s method, starting at \( x = 0 \) with two steps of equal size, to approximate \( f(-0.6) \).

   (A) -0.25
   (B) -0.55
   (C) 0.55
   (D) 0.25
   (E) -1.2
4. (calculator not allowed) Assume that \( f \) and \( f' \) have the values given in the table. Use Euler’s Method with two equal steps to approximate the value of \( f(4.4) \).

\[
\begin{array}{c|c|c|c}
\hline
x & 4 & 4.2 & 4.4 \\
\hline
f'(x) & -0.5 & -0.3 & -0.1 \\
f(x) & 2 & & \\
\hline
\end{array}
\]

(A) 1.96  
(B) 1.88  
(C) 1.84  
(D) 0.94  
(E) 0.88
Free Response

5. (calculator not allowed)
   Let \( f \) be the function satisfying \( f''(x) = -3xf(x) \), for all real numbers \( x \) with \( f(1) = 4 \).
   
   (b) Use Euler’s Method, starting at \( x = 1 \), with step size of 0.5, to approximate \( f(2) \).

6. (calculator not allowed)
   Consider the differential equation \( \frac{dy}{dx} = 3x + 2y + 1 \)
   
   (c) Let \( y = f(x) \) be a particular solution to the differential equation with initial condition \( f(0) = -2 \). Use Euler’s Method, starting at \( x = 0 \) with step size of \( \frac{1}{2} \), to approximate \( f(1) \).
   
   (d) Let \( y = g(x) \) be another solution to the differential equation with initial condition \( g(0) = k \), where \( k \) is a constant. Euler’s Method, starting at \( x = 0 \) with step size of 1, gives the approximation \( g(1) = 0 \). Find the value of \( k \).
7. (calculator not allowed)

Consider the differential equation \( \frac{dy}{dx} = 6x^2 - x^2y \). Let \( y = f(x) \) be a particular solution to this differential equation with the initial condition \( f(-1) = 2 \).

(a) Use Euler’s method with two steps of equal size, starting at \( x = -1 \), to approximate \( f(0) \). Show the work that leads to your answer.

8. (calculator not allowed)

<table>
<thead>
<tr>
<th>( x )</th>
<th>1</th>
<th>1.1</th>
<th>1.2</th>
<th>1.3</th>
<th>1.4</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f'(x) )</td>
<td>8</td>
<td>10</td>
<td>12</td>
<td>13</td>
<td>14.5</td>
</tr>
</tbody>
</table>

The function \( f \) is twice differentiable for \( x > 0 \) with \( f'(1) = 15 \) and \( f''(1) = 20 \). Values of \( f'' \), the derivative of \( f \), are given for selected values of \( x \) in the table above. (Note: all parts of question are included to observe the different approximation methods)

(a) Write an equation for the line tangent to the graph of \( f \) at \( x = 1 \). Use this line to approximate \( f(1.4) \).

(b) Use a midpoint Riemann sum with two subintervals of equal length and values from the table to approximate \( \int_1^{1.4} f'(x) \, dx \). Use the approximation for \( \int_1^{1.4} f''(x) \, dx \) to estimate the value of \( f(1.4) \). Show the computations that lead to your answer.

(c) Use Euler’s method, starting at \( x = 1 \) with two steps of equal size, to approximate \( f(1.4) \). Show the computations that lead to your answer.

(d) Write the second-degree Taylor polynomial for \( f \) about \( x = 1 \). Use the Taylor polynomial to approximate \( f(1.4) \).
Multiple Choice Logistic Growth

9. (calculator not allowed)
The number of moose in a national park is modeled by the function \( M \) that satisfies the logistic differential equation \( \frac{dM}{dt} = 0.6M \left( 1 - \frac{M}{200} \right) \), where \( t \) is the time in years and \( M(0) = 50 \). What is \( \lim_{t \to \infty} M(t) \)?

(A) 50
(B) 200
(C) 500
(D) 1000
(E) 2000

10. (calculator not allowed)

Which of the following differential equations for a population \( P \) could model the logistic growth shown in the figures above?

(A) \( \frac{dP}{dt} = 0.2P - 0.001P^2 \)
(B) \( \frac{dP}{dt} = 0.1P - 0.001P^2 \)
(C) \( \frac{dP}{dt} = 0.2P^2 - 0.001P \)
(D) \( \frac{dP}{dt} = 0.1P^2 - 0.001P \)
(E) \( \frac{dP}{dt} = 0.1P^2 + 0.001P \)
Free Response
11. (calculator not allowed)

Consider the logistic differential equation \( \frac{dy}{dt} = \frac{y}{8}(6 - y) \). Let \( y = f(t) \) be the particular solution to the differential equation with \( f(0) = 8 \).

(a) A slope field for this differential equation is given below. Sketch the possible solution curves through the points (3, 2) and (0, 8).

(b) Use Euler’s method, starting at \( t = 0 \) with two steps of equal size, to approximate \( f(1) \).
12. (calculator not allowed)

A population is modeled by a function $P$ that satisfies the logistic differential equation
\[ \frac{dP}{dt} = P \left(1 - \frac{P}{12}\right). \]

(a) If $P(0) = 3$, what is $\lim_{t \to \infty} P(t)$?
If $P(0) = 20$, what is $\lim_{t \to \infty} P(t)$?

(b) If $P(0) = 3$, for what value of $P$ is the population growing the fastest?

A certain rumor spreads through a community at the rate \( \frac{dy}{dt} = 2y(1-y) \), where \( y \) is the proportion of the population that has heard the rumor at time \( t \).

(a) What proportion of the population has heard the rumor when it is spreading the fastest?

(b) If at time \( t = 0 \) ten percent of the people have heard the rumor, find \( y \) as a function of \( t \).

(c) At what time \( t \) is the rumor spreading the fastest?