Euler's Method and Logistic Growth (BC Only)

Euler's Method

Students should be able to:

- Approximate numerical solutions of differential equations using Euler's method without a calculator.
- Recognize the method as a recursion formula extension of the point-slope version of the equation of a tangent line.
- General Format: $y - y_1 = m(x - x_1)$ (point-slope form of equation of tangent line)

  $y = y_1 + m(x - x_1)$ (where m is $\frac{dy}{dx}$=given differential equation, $x - x_1$ is the step size (h or Δx, and y_1 is the given y the first time, for other iterations it is the previous y or y_{n-1})

- Produce ordered pairs of points for the function to be approximated given an increment for the change in x.

Logistic Growth

Students should be able to:

- Know that the solution of the general logistic differential equation is

$$\frac{dP}{dt} = kP(L - P) \quad \text{is} \quad P = \frac{L}{1 + ae^{-(Lk)t}}$$

  where L is the maximum carrying capacity and k is the growth constant (L and k are both positive).
- Determine the limit of the population over a long period of time (always the maximum carrying capacity L, regardless of the initial population)
- Determine when the population is growing the fastest (always when the population is half of the maximum carrying capacity)
- Factor the differential equation (if not given in factored form) so that it is in the form of $\frac{dP}{dt} = kP(L - P)$ where L, the maximum carrying capacity, can be easily determined. L is the maximum population that can be sustained or supported as time t increases. A population that satisfies the differential equation does not grow without bound (as in exponential growth) but approaches the carrying capacity L as time t increases.

Multiple Choice Euler's Method

1. (calculator not allowed)

   Let $y = f(x)$ be the solution to the differential equation $\frac{dy}{dx} = x + y$ with the initial condition $f(1) = 2$. What is the approximation for $f(2)$ if Euler's method is used, starting at $x = 1$ with a step size of 0.5?

   (A) 3
   (B) 5
   (C) 6
   (D) 10
   (E) 12

2. (calculator not allowed)

   Given that $y(1) = -3$ and $\frac{dy}{dx} = 2x + y$, what is the approximation for $y(2)$ if Euler's method is used with a step size of 0.5, starting at $x = 1$?

   (A) -5
   (B) -4.25
   (C) -4
   (D) -3.75
   (E) -3.5

3. (calculator not allowed)

   Let $y = f(x)$ be the particular solution to the differential equation $\frac{dy}{dx} = x + 2y$ with the initial condition $f(0) = 1$. Use Euler's method, starting at $x = 0$ with two steps of equal size, to approximate $f(-0.6)$.

   (A) -0.25
   (B) -0.55
   (C) 0.55
   (D) 0.25
   (E) -1.2

4. (calculator not allowed)
Assume that f and f' have the values given in the table. Use Euler's Method with two equal steps to approximate the value of $f(4.4)$.

x	4	4.2	4.4
$f'(x)$	–0.5	–0.3	–0.1
$f(x)$	2		

(A) 1.96
(B) 1.88
(C) 1.84
(D) 0.94
(E) 0.88

Free Response

5. (calculator not allowed)
Let f be the function satisfying $f'(x) = -3xf(x)$, for all real numbers x with $f(1) = 4$.

(b) Use Euler's Method, starting at $x = 1$, with step size of 0.5, to approximate $f(2)$.

6. (calculator not allowed)
Consider the differntial equation $\frac{dy}{dx} = 3x + 2y + 1$

(c) Let $y = f(x)$ be a particular solution to the differential equation with intial condition $f(0) = -2$. Use Euler's Method, starting at $x = 0$ with step size of $\frac{1}{2}$, to approximate $f(1)$.

(d) Let $y = g(x)$ be another solution to the differential equation with initial condition $g(0) = k$, where k is a constant. Euler's Method, starting at $x = 0$ with step size of 1, gives the approximation $g(1) = 0$. Find the value of k.

7. (calculator not allowed)

Consider the differential equation $\frac{dy}{dx} = 6x^2 - x^2 y$. Let $y = f(x)$ be a particular solution to this differential equation with the initial condition $f(-1) = 2$.

(a) Use Euler's method with two steps of equal size, starting at $x = -1$, to approximate $f(0)$. Show the work that leads to your answer.

8. (calculator not allowed)

x	1	1.1	1.2	1.3	1.4
$f'(x)$	8	10	12	13	14.5

The function f is twice differentiable for $x > 0$ with $f(1) = 15$ and $f''(1) = 20$. Values of f', the derivative of f, are given for selected values of x in the table above. (Note: all parts of question are included to observe the different approximation methods)

(a) Write an equation for the line tangent to the graph of f at $x = 1$. Use this line to approximate $f(1.4)$.

(b) Use a midpoint Riemann sum with two subintervals of equal length and values from the table to approximate $\int_1^{1.4} f'(x)\,dx$. Use the approximation for $\int_1^{1.4} f'(x)\,dx$ to estimate the value of $f(1.4)$. Show the computations that lead to your answer.

(c) Use Euler's method, starting at $x = 1$ with two steps of equal size, to approximate $f(1.4)$. Show the computations that lead to your answer.

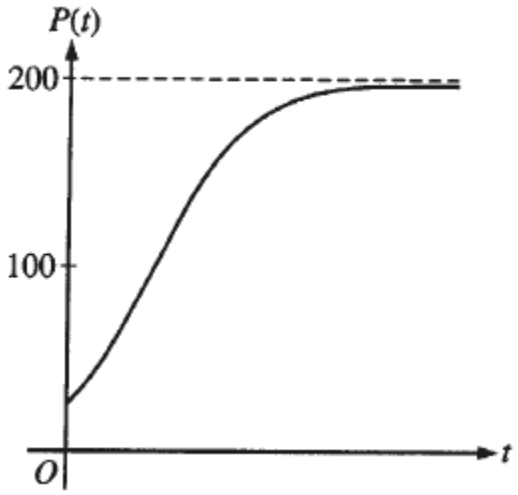
(d) Write the second-degree Taylor polynomial for f about $x = 1$. Use the Taylor polynomial to approximate $f(1.4)$.

Multiple Choice Logistic Growth

9. (calculator not allowed)
The number of moose in a national park is modeled by the function M that satisfies the logistic differential equation $\frac{dM}{dt} = 0.6M\left(1 - \frac{M}{200}\right)$, where t is the time in years and $M(0) = 50$. What is $\lim_{t \to \infty} M(t)$?

(A) 50
(B) 200
(C) 500
(D) 1000
(E) 2000

10. (calculator not allowed)



Which of the following differential equations for a population P could model the logistic growth shown in the figues above?

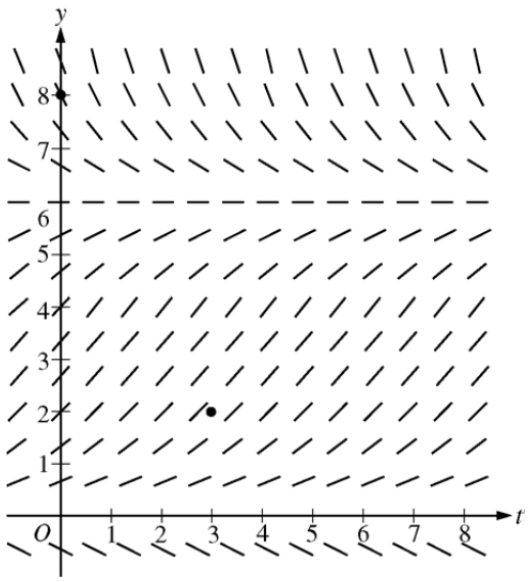
(A) $\frac{dP}{dt} = 0.2P - 0.001P^2$
(B) $\frac{dP}{dt} = 0.1P - 0.001P^2$
(C) $\frac{dP}{dt} = 0.2P^2 - 0.001P$
(D) $\frac{dP}{dt} = 0.1P^2 - 0.001P$
(E) $\frac{dP}{dt} = 0.1P^2 + 0.001P$

Free Response

11. (calculator not allowed)

Consider the logistic differential equation $\frac{dy}{dt} = \frac{y}{8}(6 - y)$. Let $y = f(t)$ be the particular solution to the differential equation with $f(0) = 8$.

(a) A slope field for this differential equation is given below. Sketch the possible solution curves through the points (3, 2) and (0, 8).



(b) Use Euler's method, starting at $t = 0$ with two steps of equal size, to approximate $f(1)$.

12. (calculator not allowed)

A population is modeled by a function P that satisfies the logistic differential equation $\frac{dP}{dt} = \frac{P}{5}\left(1 - \frac{P}{12}\right)$.

(a) If $P(0) = 3$, what is $\lim_{t \to \infty} P(t)$?

If $P(0) = 20$, what is $\lim_{t \to \infty} P(t)$?

(b) If $P(0) = 3$, for what value of P is the population growing the fastest?

13. (1991 BC6)

A certain rumor spreads through a community at the rate $\frac{dy}{dt} = 2y(1-y)$, where y is the proportion of the population that has heard the rumor at time t.

(a) What proportion of the population has heard the rumor when it is spreading the fastest?

(b) If at time $t = 0$ ten percent of the people have heard the rumor, find y as a function of t.

(c) At what time t is the rumor spreading the fastest?