

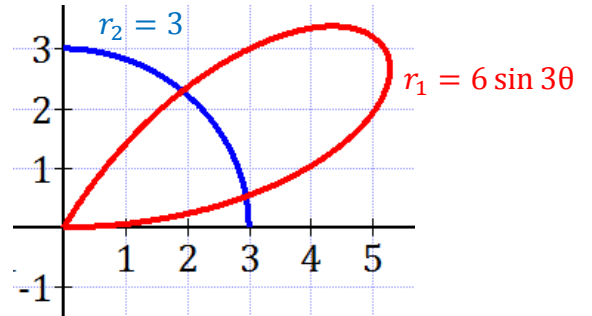


Derivatives and Equations in Polar Coordinates

1. The graphs of the polar curves $r_1 = 6 \sin 3\theta$ and $r_2 = 3$ are shown to the right.

(You may use your calculator for all sections of this problem.)

- a) Find the coordinates of the points of intersection of both curves for $0 \leq \theta < \frac{\pi}{2}$. Write your answers using polar coordinates.
- b) Write the coordinates of the points of intersection using now rectangular coordinates.

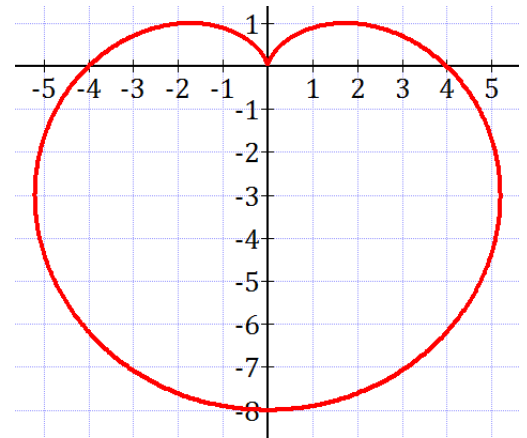


- c) Find $\left. \frac{dr_1}{d\theta} \right|_{\theta = \frac{\pi}{4}}$. Interpret the meaning of your answer in the context of the problem.
- d) For $0 \leq \theta < \frac{\pi}{2}$, there are two points on r_1 with x-coordinate equal to 4. Find the subject points. Express your answer using polar coordinates.
- e) Write in terms of θ an expression for $\frac{dy}{dx}$, the slope of the tangent line to the graph of r_1 .
- f) Write in terms of x and y an equation for the line tangent to the graph of the curve r_1 at the point where $\theta = \frac{\pi}{4}$.

2. The graph of the polar curve $r = 4 - 4 \sin \theta$ is shown to the right.

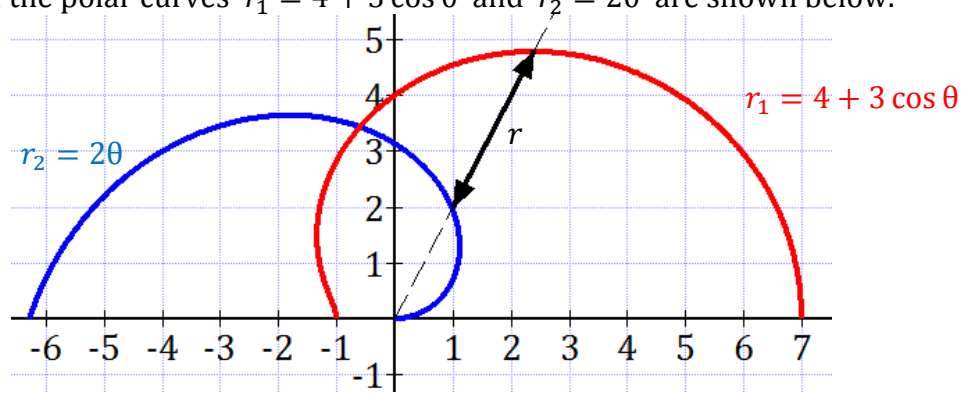
(You may use your calculator for all sections of this problem.)

- a) For $0 \leq \theta < 2\pi$, there are two points on r with y-coordinate equal to -4 . Find the subject points. Express your answers using polar coordinates.
- b) Write an expression for the x-coordinate of each point on the graph of $r = 4 - 4 \sin \theta$. Express your answer in terms of θ .
- c) A particle moves along the polar curve $r = 4 - 4 \sin \theta$ so that at time t seconds, $\theta = t^2$. Find the time t in the time interval $1 \leq t \leq 2$ for which the x-coordinate of the particle's position is -1 .



- d) Find $\left. \frac{dr}{dt} \right|_{t=2}$. Interpret the meaning of your answer in the context of the problem.
- e) Find $\left. \frac{dx}{dt} \right|_{t=2}$. Interpret the meaning of your answer in the context of the problem.

3. The graphs of the polar curves $r_1 = 4 + 3 \cos \theta$ and $r_2 = 2\theta$ are shown below.



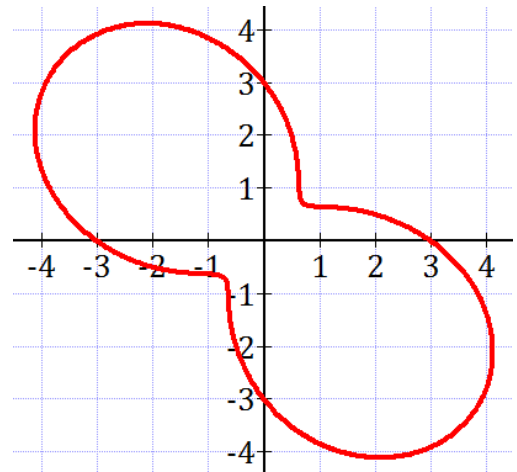
(Do NOT use your calculator for this problem unless indicated!)

- Find the coordinates of the point of intersection of both curves for $0 \leq \theta < \pi$. Write your answer using polar coordinates. (You may use your calculator for this section.)
- As the curves are traced, the distance between them, $r(\theta)$, changes (see drawing.) Find an expression for $r(\theta)$ the distance between both curves in the interval $0 \leq \theta \leq \frac{\pi}{2}$.
- Write in terms of θ an expression for $\frac{dr}{d\theta}$. Use your answer to find $\left. \frac{dr}{d\theta} \right|_{\theta = \frac{\pi}{3}}$. Interpret the meaning of your answer in the context of the problem.
- Write in terms of θ an expression for $\frac{dy}{dx}$, the slope of the tangent line to the graph of r_2 .
- Find the coordinates of the point where curve r_2 has a horizontal tangent line in the interval $0 < \theta < \pi$. Write your answer using rectangular coordinates. (You may use your calculator for this section.)

4. The graph of the polar curve $r = 3 - 2 \sin(2\theta)$ for $0 \leq \theta < 2\pi$ is shown to the right.

(You may use your calculator for all sections of this problem.)

- Write in terms of θ an expression for $\frac{dy}{dx}$, the slope of the tangent line to the graph of r .
- Find the coordinates of the point where curve r has a vertical tangent line in the interval $0 \leq \theta < \pi$. Write your answer using polar coordinates.
- Write in terms of x and y an equation for the line tangent to the graph of the curve r at the point where $\theta = \frac{\pi}{6}$.
- A particle moves along the polar curve $r = 3 - 2 \sin(2\theta)$ so that $\frac{d\theta}{dt} = 2$ for all times $t \geq 0$. Find the value of $\frac{dr}{dt}$ at $\theta = \frac{\pi}{6}$. Interpret the meaning of your answer in the context of the problem.
- Assume now that for the particle whose motion was described in section (d) we have $\theta = 2t$. Find the position vector of the particle $\langle x(t), y(t) \rangle$ in terms of t . Use your calculator to find the velocity vector and the speed of the particle at $t = 1.5$.





Derivatives and Equations in Polar Coordinates

1. The graphs of the polar curves $r_1 = 6 \sin 3\theta$ and $r_2 = 3$ are shown to the right.

(You may use your calculator for all sections of this problem.)

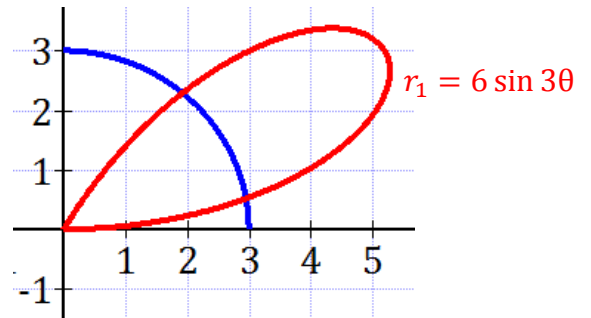
- a) Find the coordinates of the points of intersection of both curves for $0 \leq \theta < \frac{\pi}{2}$. Write your answers using polar coordinates.

Points of intersection are collision points:

$$6 \sin 3\theta = 3 \rightarrow \theta = \frac{\pi}{18} \text{ and } \frac{5\pi}{18}$$

Or $\theta \approx 0.1745$ and 0.8726

$$r = 3 \rightarrow (3, 0.1745) \text{ and } (3, 0.8726)$$



- b) Write the coordinates of the points of intersection using now rectangular coordinates.

$$(3, 0.1745) \rightarrow \begin{cases} x = r \cdot \cos \theta = 2.954 \\ y = r \cdot \sin \theta = 0.5209 \end{cases} \rightarrow (2.954, 0.5209)$$

$$(3, 0.8726) \rightarrow \begin{cases} x = r \cdot \cos \theta = 1.928 \\ y = r \cdot \sin \theta = 2.298 \end{cases} \rightarrow (1.928, 2.298)$$

- c) Find $\left. \frac{dr_1}{d\theta} \right|_{\theta=\frac{\pi}{4}}$. Interpret the meaning of your answer in the context of the problem.

By hand: $\frac{dr_1}{d\theta} = 18 \cos 3\theta \rightarrow \left. \frac{dr_1}{d\theta} \right|_{\theta=\frac{\pi}{4}} = -9\sqrt{2}$

Using a calculator: $\left. \frac{d}{d\theta} (6 \sin 3\theta) \right|_{\theta=\frac{\pi}{4}} \approx -12.7279$

When the graph of $r_1 = 6 \sin 3\theta$ is traced at $\theta = \frac{\pi}{4}$ radians the distance to the pole is decreasing at a rate equal to 12.7279 units per radian.

- d) For $0 \leq \theta < \frac{\pi}{2}$, there are two points on r_1 with x-coordinate equal to 4. Find the subject points. Express your answer using polar coordinates.

$$x = r_1 \cdot \cos \theta = 6 \sin 3\theta \cdot \cos \theta = 4 \rightarrow \theta \approx 0.253 \text{ and } 0.696$$

$$\theta \approx 0.253 \rightarrow r_1 = 6 \sin(3(0.253)) = 4.1317 \rightarrow (4.137, 0.253)$$

$$\theta \approx 0.696 \rightarrow r_1 = 6 \sin(3(0.696)) = 5.213 \rightarrow (5.213, 0.696)$$

- e) Write in terms of θ an expression for $\frac{dy}{dx}$, the slope of the tangent line to the graph of r_1 .

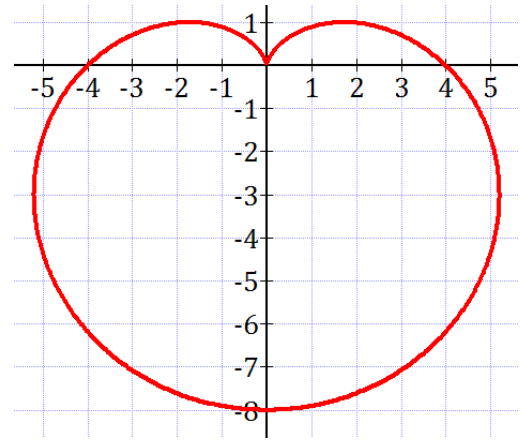
$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{3 \cos 3\theta \sin \theta + \sin 3\theta \cos \theta}{3 \cos 3\theta \cos \theta - \sin 3\theta \sin \theta}$$

- f) Write in terms of x and y an equation for the line tangent to the graph of the curve r_1 at the point where $\theta = \frac{\pi}{4}$.

$$\left. \frac{dy}{dx} \right|_{\theta=\frac{\pi}{4}} = \frac{1}{2}$$

$$\begin{cases} x = r_1 \cdot \cos \theta = 3 \\ y = r_1 \cdot \sin \theta = 3 \end{cases} \rightarrow y - 3 = \frac{1}{2}(x - 3)$$

2. The graph of the polar curve $r = 4 - 4 \sin \theta$ is shown to the right.



(You may use your calculator for all sections of this problem.)

- a) For $0 \leq \theta < 2\pi$, there are two points on r with y-coordinate equal to -4 . Find the subject points. Express your answers using polar coordinates.

$$\begin{aligned} y &= r \cdot \sin \theta = (4 - 4 \sin \theta) \sin \theta = -4 \\ &\rightarrow \theta \approx 3.8078 \quad \text{and} \quad 5.6169 \\ \theta \approx 3.8078 &\rightarrow r = 4 - 4 \sin 3.8078 = 6.472 \\ &\rightarrow (6.472, 3.8078) \\ \theta \approx 5.6169 &\rightarrow r = 4 - 4 \sin 5.6169 = 6.472 \\ &\rightarrow (6.472, 5.6169) \end{aligned}$$

- b) Write an expression for the x-coordinate of each point on the graph of $r = 4 - 4 \sin \theta$. Express your answer in terms of θ .

$$x = r \cdot \cos \theta = (4 - 4 \sin \theta) \cos \theta$$

- c) A particle moves along the polar curve $r = 4 - 4 \sin \theta$ so that at time t seconds, $\theta = t^2$. Find the time t in the time interval $1 \leq t \leq 2$ for which the x-coordinate of the particle's position is -1 .

$$x = (4 - 4 \sin t^2) \cos t^2 = -1 \rightarrow t \approx 1.5536$$

- d) Find $\left. \frac{dr}{dt} \right|_{t=2}$. Interpret the meaning of your answer in the context of the problem.

$$\begin{aligned} r &= 4 - 4 \sin t^2 \\ \text{By hand: } \frac{dr}{dt} &= -8t \cos t^2 \rightarrow \left. \frac{dr}{dt} \right|_{t=2} = -16 \cos 4 \\ \text{Using a calculator: } &\left. \frac{d}{dt} (4 - 4 \sin t^2) \right|_{t=2} \approx 10.458 \end{aligned}$$

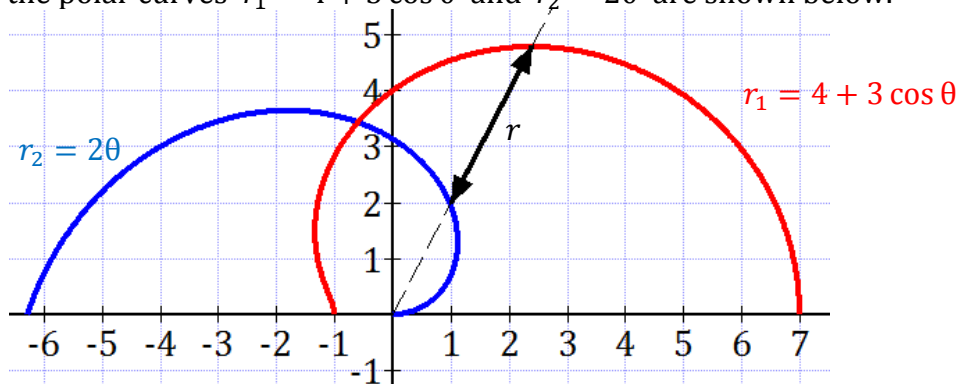
As the particle moves on the graph of $r = 4 - 4 \sin \theta$, when $t = 2$ seconds the distance to the pole is increasing at a rate equal to 10.458 units per second.

- e) Find $\left. \frac{dx}{dt} \right|_{t=2}$. Interpret the meaning of your answer in the context of the problem.

$$\text{Using a calculator: } \left. \frac{d}{dt} ((4 - 4 \sin t^2) \cos t^2) \right|_{t=2} \approx 14.4368$$

As the particle moves on the graph of $r = 4 - 4 \sin \theta$, when $t = 2$ seconds the particle moves to the right with a horizontal speed equal to 14.4368 units per second.

3. The graphs of the polar curves $r_1 = 4 + 3 \cos \theta$ and $r_2 = 2\theta$ are shown below.



(Do NOT use your calculator for this problem unless indicated!)

- a) Find the coordinates of the point of intersection of both curves for $0 \leq \theta < \pi$. Write your answer using polar coordinates. (You may use your calculator for this section.)

The point of intersection is also a collision point:

$$4 + 3 \cos \theta = 2\theta \rightarrow \theta \approx 1.7429$$

$$r = 2(1.7429) = 3.4859 \rightarrow (3.4859, 1.7429)$$

- b) As the curves are traced, the distance between them, $r(\theta)$, changes (see drawing.) Find an expression for $r(\theta)$ the distance between both curves in the interval $0 \leq \theta \leq \frac{\pi}{2}$.

$$r(\theta) = r_1 - r_2 = 4 + 3 \cos \theta - 2\theta$$

- c) Write in terms of θ an expression for $\frac{dr}{d\theta}$. Use your answer to find $\left. \frac{dr}{d\theta} \right|_{\theta=\frac{\pi}{3}}$. Interpret the meaning of your answer in the context of the problem.

$$\frac{dr}{d\theta} = -3 \sin \theta - 2 \rightarrow \left. \frac{dr}{d\theta} \right|_{\theta=\frac{\pi}{3}} = -\frac{3\sqrt{3}}{2} - 2$$

When the graphs of $r_1 = 6 \sin 3\theta$ and $r_2 = 2\theta$ are traced at $\theta = \frac{\pi}{3}$ radians the distance between the two graphs is decreasing at a rate equal to $-\frac{3\sqrt{3}}{2} - 2$ units per radian.

- d) Write in terms of θ an expression for $\frac{dy}{dx}$, the slope of the tangent line to the graph of r_2 .

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\sin \theta + \theta \cos \theta}{\cos \theta - \theta \sin \theta}$$

- e) Find the coordinates of the point where curve r_2 has a horizontal tangent line in the interval $0 < \theta < \pi$. Write your answer using rectangular coordinates. (You may use your calculator for this section.)

$$\frac{dy}{dx} = 0 \rightarrow \sin \theta + \theta \cos \theta = 0 \rightarrow \theta \approx 2.0287$$

$$r = 2(2.0287) = 4.0575$$

$$\begin{cases} x = r \cdot \cos \theta = -1.7939 \\ y = r \cdot \sin \theta = 3.639 \end{cases} \rightarrow (-1.7939, 3.639)$$

4. The graph of the polar curve $r = 3 - 2 \sin(2\theta)$ for $0 \leq \theta < 2\pi$ is shown to the right.

(You may use your calculator for all sections of this problem.)

- a) Write in terms of θ an expression for $\frac{dy}{dx}$, the slope of the tangent line to the graph of r .

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{-4 \cos 2\theta \sin \theta + (3 - 2 \sin(2\theta)) \cos \theta}{-4 \cos 2\theta \cos \theta - (3 - 2 \sin(2\theta)) \sin \theta}$$

- b) Find the coordinates of the point where curve r has a vertical tangent line in the interval $0 \leq \theta < \pi$. Write your answer using polar coordinates.

$$\frac{dy}{dx} \text{ is undefined} \rightarrow -4 \cos 2\theta \cos \theta - (3 - 2 \sin(2\theta)) \sin \theta = 0 \rightarrow \theta \approx 2.670$$

$$r = 3 - 2 \sin(2(2.670)) = 4.6177 \rightarrow (4.6177, 2.670)$$

- c) Write in terms of x and y an equation for the line tangent to the graph of the curve r at the point where $\theta = \frac{\pi}{6}$.

$$\left. \frac{dy}{dx} \right|_{\theta=\frac{\pi}{6}} \approx -0.041 \quad (\text{by hand: } \left. \frac{dy}{dx} \right|_{\theta=\frac{\pi}{6}} = \frac{3\sqrt{3}-5}{-3-\sqrt{3}})$$

$$\left. \begin{aligned} x &= r \cdot \cos \theta = \frac{3\sqrt{3}-3}{2} = 1.098 \\ y &= r \cdot \sin \theta = \frac{3-\sqrt{3}}{2} = 0.6339 \end{aligned} \right\} \rightarrow y - 0.6339 = -0.041(x - 1.098)$$

- d) A particle moves along the polar curve $r = 3 - 2 \sin(2\theta)$ so that $\frac{d\theta}{dt} = 2$ for all times $t \geq 0$.

Find the value of $\frac{dr}{dt}$ at $\theta = \frac{\pi}{6}$. Interpret the meaning of your answer in the context of the problem.

$$\begin{aligned} \frac{dr}{dt} &= \frac{dr}{d\theta} \cdot \frac{d\theta}{dt} = (-4 \cos(2\theta))(2) \\ \left. \frac{dr}{dt} \right|_{\theta=\frac{\pi}{6}} &= -4 \end{aligned}$$

As the particle moves on the graph of $r = 3 - 2 \sin(2\theta)$, when it is at the point where $\theta = \frac{\pi}{6}$ radians the distance to the pole is decreasing at a rate equal to 4 units per second.

- e) Assume now that for the particle whose motion was described in section (d) we have $\theta = 2t$. Find the position vector of the particle $\langle x(t), y(t) \rangle$ in terms of t . Use your calculator to find the velocity vector and the speed of the particle at $t = 1.5$.

$$\left. \begin{aligned} x &= r \cdot \cos \theta = (3 - 2 \sin(4t)) \cos(2t) \\ y &= r \cdot \sin \theta = (3 - 2 \sin(4t)) \sin(2t) \end{aligned} \right\} \rightarrow \langle (3 - 2 \sin(4t)) \cos(2t), (3 - 2 \sin(4t)) \sin(2t) \rangle$$

$$\text{Velocity vector: } \left\langle \left. \frac{dx}{dt} \right|_{t=1.5}, \left. \frac{dy}{dt} \right|_{t=1.5} \right\rangle = \langle 6.600, -8.130 \rangle$$

$$\text{Speed: } \sqrt{\left(\left. \frac{dx}{dt} \right|_{t=1.5} \right)^2 + \left(\left. \frac{dy}{dt} \right|_{t=1.5} \right)^2} = 10.472$$

