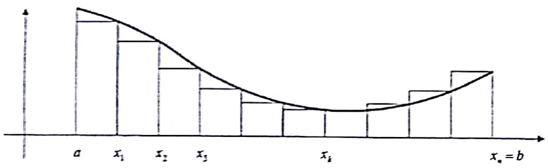
AP Calculus AB   Name: Key

The Riemann Sum and the Definite Integral

We begin our introduction to the Riemann Sum by considering non-negative functions which are continuous over an interval $[a.b]$. To simplify the explanation and the calculations, the interval $[a.b]$ will be divided into subintervals of equal width. and the sample points will correspond to the right endpoints of the subintervals. A more general/rigorous treatment of the Riemann Sum may be found in the calculus textbook used by Pure and Applied Science students.

Let the non-negative function $y = f(x)$ be continuous over $[a.b]$. We divide $[a.b]$ into n equal subintervals of width $\Delta x = \frac{b-a}{n}$. The right endpoints of the subintervals are designated $x_1, x_2, x_3, \ldots, x_k, \ldots, x_n$ where $x_k = a + k\Delta x$ and where $x_n = b$. For each subinterval we construct a rectangle as shown in the diagram.



The base of each rectangle is Δx. The height of rectangle k (the rectangle on the subinterval with x_k as right endpoint) is $f(x_k)$. It follows that the area of rectangle k is $f(x_k)\Delta x$. The sum of the areas of all n rectangles is called the **Riemann Sum**. I.e. the Riemann Sum is equal to the expression $\sum_{k=1}^{n} f(x_k)\Delta x$. We see that the Riemann Sum is an approximation of the exact area under the graph of f from a to b. The larger the value of n the better the approximation. It can be proven that the limit at infinity of the Riemann Sum is the exact area under the graph of f from a to b. This limit has a special name and notation. It is called the **definite integral**.

Definition of Definite Integral If f is a continuous function defined on $[a.b]$, and if $[a.b]$ is divided into n equal subintervals of width $\Delta x = \frac{b-a}{n}$, and if $x_k = a + k\Delta x$ is the right endpoint of subinterval k, then the definite integral of f from a to b is the number

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\Delta x$$

1. width $= \Delta x = \frac{b-a}{n}$
2. $x_k = a + k \cdot \Delta x$
3. height $= f(x_k)$
4. Area Rect $= f(x_k) \cdot \Delta x$

Practice: Use the definition of the definite integral to write each integral using limit notation.

1. $\int_0^5 4x\,dx$

$a=0$
$b=5$

width $= \Delta x = \frac{5-0}{n} = \frac{5}{n}$

$x_k = 0 + k \cdot \frac{5}{n} = \frac{5k}{n}$

$f(x_k) = 4\left(\frac{5k}{n}\right) = \frac{20k}{n}$

Area $= \frac{5}{n} \cdot \frac{20k}{n} = \frac{100k}{n^2}$

$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{100k}{n^2}$

2. $\int_0^2 (x^2+10)\,dx$

$a=0$
$b=2$

width $= \Delta x = \frac{2}{n}$

$x_k = 0 + k \cdot \frac{2}{n} = \frac{2k}{n}$

$f(x_k) = \left(\frac{2k}{n}\right)^2 + 10 = \frac{4k^2}{n^2} + 10$

Area $= \frac{2}{n}\left(\frac{4k^2}{n^2} + 10\right)$

$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{2}{n}\left(\frac{4k^2}{n^2} + 10\right)$

3. $\int_0^3 (5x^2 - 2x)\,dx$

$a=0$
$b=3$

width $= \frac{3}{n} = \Delta x$

$x_k = 0 + k \cdot \frac{3}{n} = \frac{3k}{n}$

$f(x_k) = 5\left(\frac{3k}{n}\right)^2 - 2\left(\frac{3k}{n}\right)$

$= \frac{45k^2}{n^2} - \frac{6k}{n}$

Area $= \left(\frac{45k^2}{n^2} - \frac{6k}{n}\right) \cdot \frac{3}{n}$

$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{3}{n}\left(\frac{45k^2}{n^2} - \frac{6k}{n}\right)$

AP Practice Problems: $4(30)+5(34)+3(28)$

Part A. No Calculator Allowed.

x	2	5	10	14
$f(x)$	12	28	34	30

1. The function f is continuous on the closed interval [2, 14] and has values as shown in the table above. Using the subintervals [2, 5], [5, 10], and [10, 14], what is the approximation of $\int_2^{14} f(x)dx$ found by using a right Riemann sum?

(A) 296 (B) 312 (C) 343 (D) 374 (E) 390

2. Which of the following limits is equal to $\int_3^5 x^4\,dx$?

(A) $\lim_{n\to\infty}\sum_{k=1}^{n}\left(3+\frac{k}{n}\right)^4\frac{1}{n}$

(B) $\lim_{n\to\infty}\sum_{k=1}^{n}\left(3+\frac{k}{n}\right)^4\frac{2}{n}$

(C) $\lim_{n\to\infty}\sum_{k=1}^{n}\left(3+\frac{2k}{n}\right)^4\frac{1}{n}$

(D) $\lim_{n\to\infty}\sum_{k=1}^{n}\left(3+\frac{2k}{n}\right)^4\frac{2}{n}$

$\Delta X = \frac{2}{n}$

$X_K = 3 + \frac{2K}{n}$

$f(X_K) = \left(3+\frac{2K}{n}\right)^4$

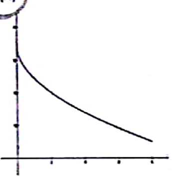
$\text{Area} = \left(3+\frac{2K}{n}\right)^4 \cdot \frac{2}{n}$

Part B. Graphing Calculator Allowed.

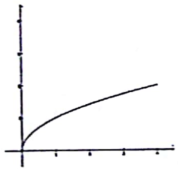
3. If a trapezoidal sum overapproximates $\int_0^4 f(x)dx$, and a right Riemann sum underapproximates $\int_0^4 f(x)dx$, which of the following could be the graph of $y = f(x)$?

TAM over CC↑
RRAM under dec

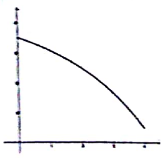
(A)



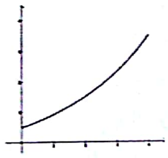
(B)



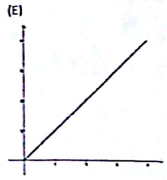
(C)



(D)



(E)



x	2	5	7	8
$f(x)$	10	30	40	20

4. The function f is continuous on the closed interval [2, 8] and has values that are given in the table above. Using the subintervals [2, 5], [5, 7], and [7, 8], what is the trapezoidal approximation of $\int_2^8 f(x)dx$?

(A) 110 (B) 130 (C) 160 (D) 190 (E) 210

$\frac{1}{2}\left(3(10+30)+2(30+40)+1(40+20)\right)$

t (sec)	0	2	4	6
$a(t)$ (ft/sec^2)	5	2	8	3

5. The data for the acceleration $a(t)$ of a car from 0 to 6 seconds are given in the table above. If the velocity at $t = 0$ is 11 feet per second, the approximate value of the velocity at $t = 6$, computed using a left-hand Riemann sum with three subintervals of equal length, is

(A) 26 ft/sec (B) 30 ft/sec (C) 37 ft/sec (D) 39 ft/sec (E) 41 ft/sec

$2(5+2+8)$

$2(15)=30+11=41$

x	0	0.5	1.0	1.5	2.0
$f(x)$	3	3	5	8	13

6. A table of values for a continuous function f is shown above. If four equal subintervals of [0, 2] are used, which of the following is the trapezoidal approximation of $\int_0^2 f(x)dx$?

(A) 8 (B) 12 (C) 16 (D) 24 (E) 32

$\frac{1}{2}\cdot\frac{1}{2}\left((3+3)+(3+5)+(5+8)+8+13)\right)$

$=12$