1969 AB3/BC3

Given $f(x) = \frac{1}{x} + \ln x$, defined only on the closed interval $\frac{1}{e} \le x \le e$.

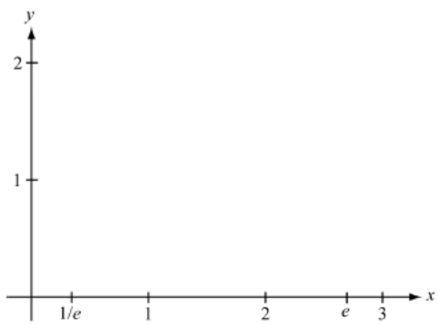
(a) Showing your reasoning, determine the value of x at which f has its

(i) absolute maximum,
(ii) absolute minimum.

(b) For what values of x is the curve concave up?

(c) On the coordinate axes provided, sketch the graph of f over the interval $\frac{1}{e} \le x \le e$.

(d) Given that the mean value (average ordinate) of f over the interval is $\frac{2}{e-1}$, state in words a geometrical interpretation of this number relative to the graph.



1970 AB3/BC2

Consider the function f given by $f(x) = x^{\frac{4}{3}} + 4x^{\frac{1}{3}}$ on the interval $-8 \le x \le 8$.

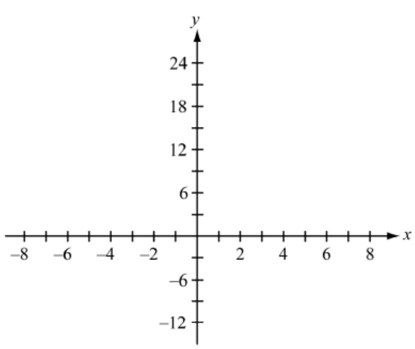
(a) Find the coordinates of all points at which the tangent to the curve is a horizontal line.

(b) Find the coordinates of all points at which the tangent to the curve is a vertical line.

(c) Find the coordinates of all points at which the absolute maximum and absolute minimum occur.

(d) For what values of x is this function concave down?

(e) On the axes provided, sketch the graph of the function on this interval.



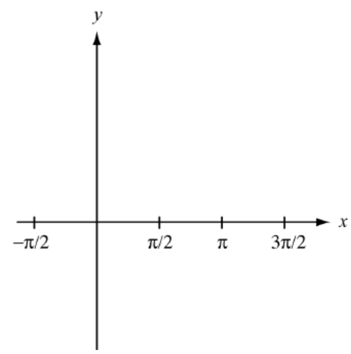
1975 AB4/BC1

Given the function defined by $y = x + \sin x$ for all x such that $-\frac{\pi}{2} \le x \le \frac{3\pi}{2}$.

(a) Find the coordinates of <u>all</u> maximum and minimum points on the given interval. Justify your answers.

(b) Find the coordinates of <u>all</u> points of inflection on the given interval. Justify your answers.

(c) On the axes provided, sketch the graph of the function.



1977 AB2

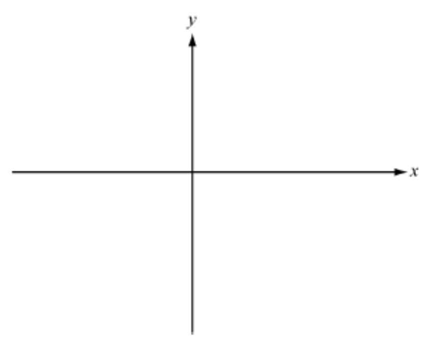
Consider the function f defined by $f(x) = (x^2 - 1)^3$ for all real numbers x.

(a) For what values of x is the function increasing?

(b) Find the x- and y-coordinates of the relative maximum and minimum points. Justify your answer.

(c) For what values of x is the graph of f concave upward?

(d) Using the information found in parts (a), (b), and (c), sketch the graph of f on the axes provided.



Answers

1969 AB3/BC3
Solution

(a) $f'(x) = -\frac{1}{x^2} + \frac{1}{x} = \frac{x-1}{x^2}$

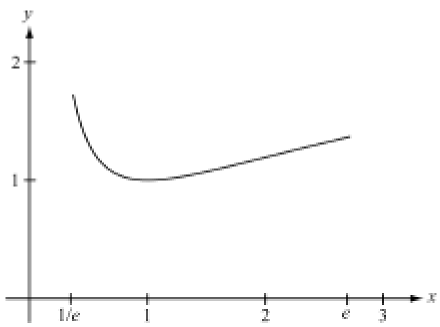
$f'(x) = 0$ at $x = 1$.

The three candidates are
$f(1) = 1$
$f\left(\frac{1}{e}\right) = e - 1$
$f(e) = \frac{1}{e} + 1 = \frac{e+1}{e}$
Therefore the absolute maximum is at $x = \frac{1}{e}$ and the absolute minimum is at $x = 1$.

(b) $f''(x) = \frac{2}{x^3} - \frac{1}{x^2} = \frac{2-x}{x^3} > 0$ for $\frac{1}{e} \le x < 2$.

Therefore the curve is concave up for $\frac{1}{e} \le x < 2$.

(c)



(d) $\frac{2}{e-1}$ is the height of a rectangle of width $\left(e - \frac{1}{e}\right)$ and having the same area as that enclosed by the graph of $y = f(x)$, the vertical lines $x = \frac{1}{e}$ and $x = e$, and the x-axis.

1970 AB3/BC2
Solution

(a) $f(x) = x^{4/3} + 4x^{1/3} = x^{1/3}(x+4)$

$f'(x) = \frac{4}{3}x^{1/3} + \frac{4}{3}x^{-2/3} = \frac{4}{3}\left(\frac{x+1}{x^{2/3}}\right)$

$f'(x) = 0$ at $x = -1$. There is a horizontal tangent at $(-1, -3)$.

(b) There is a vertical tangent at $(0, 0)$.

(c) The absolute maximum and absolute minimum must occur at a critical point or an endpoint. The candidates are

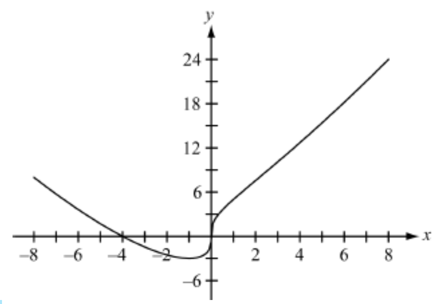
$(-8, 8)$, $(-1, -3)$, $(0, 0)$, and $(8, 24)$

So the absolute maximum is at $(8, 24)$ and the absolute minimum is at $(-1, -3)$.

(d) $f''(x) = \frac{4}{9}x^{-2/3} - \frac{8}{9}x^{-5/3} = \frac{4}{9}\left(\frac{x-2}{x^{5/3}}\right)$

The graph is concave down for $0 < x < 2$.

(e)



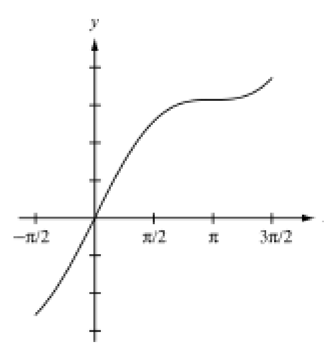
1975 AB4/BC1
Solution

(a) $y' = 1 + \cos x$
Therefore $x = \pi$ is the only critical point on the interval $-\frac{\pi}{2} \le x \le \frac{3\pi}{2}$. But $y' \ge 0$ on this interval, hence π is not an extreme point. The minimum and maximum must occur at the endpoints.
At $x = -\frac{\pi}{2}$, $y = -\frac{\pi}{2} + \sin\left(-\frac{\pi}{2}\right) = -\frac{\pi}{2} - 1$
At $x = \frac{3\pi}{2}$, $y = \frac{3\pi}{2} + \sin\left(\frac{3\pi}{2}\right) = \frac{3\pi}{2} - 1$
The absolute minimum is at $\left(-\frac{\pi}{2}, -\frac{\pi}{2} - 1\right)$.

The absolute maximum is at $\left(\frac{3\pi}{2}, \frac{3\pi}{2} - 1\right)$.

(b) $y'' = -\sin x$
$y'' = 0$ at $x = 0$ and $x = \pi$.
$y'' > 0$ for $-\frac{\pi}{2} < x < 0$
$y'' < 0$ for $0 < x < \pi$
$y'' > 0$ for $\pi < x < \frac{3\pi}{2}$
Therefore $(0, 0)$ and (π, π) are inflection points.

(c)



1977 AB2
Solution

(a) $f(x)=(x^2-1)^3$
$f'(x)=6x(x^2-1)^2$
$x<0 \Rightarrow f'(x)<0$
$x>0 \Rightarrow f'(x)>0$

Therefore the function is increasing for $x>0$.

(b) $f'(x)=6x(x^2-1)^2=0$
$x=0, x=1, x=-1$

Since f is decreasing for $x<0$ and increasing for $x>0$, the only relative minimum point is at $x=0, y=-1$ and there are no relative maximum points.

(c) $f''(x)=6(x^2-1)^2+24x^2(x^2-1)=6(x^2-1)(5x^2-1)$
$f''(x)=0$ for $x=1, x=-1, x=\sqrt{\frac{1}{5}}, x=-\sqrt{\frac{1}{5}}$.

The graph of f is concave up when $6(x^2-1)(5x^2-1)>0$. This happens for all x in $(-\infty,-1)\cup\left(-\sqrt{\frac{1}{5}},\sqrt{\frac{1}{5}}\right)\cup(1,\infty)$

(d)