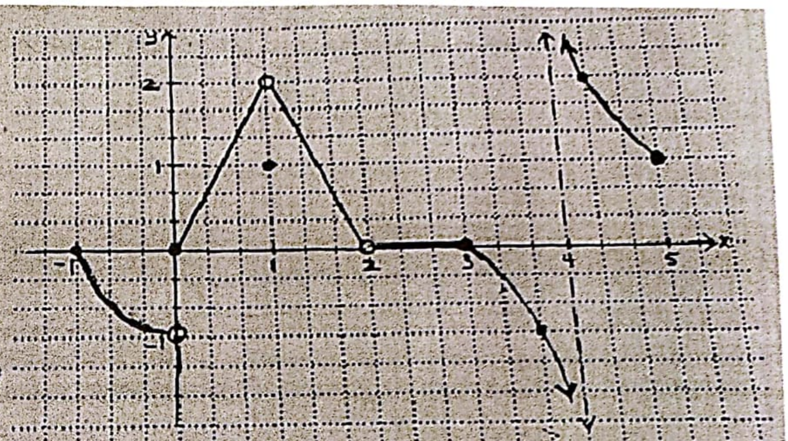
Key

What makes a function continuous?

The Meaning of Continuity:

Consider the function f defined below:

$$f(x)=\begin{cases} x^2-1, & -1\le x<0 \\ 2x, & 0\le x<1 \\ 1, & x=1 \\ -2x+4, & 1<x<2 \\ 0, & 2<x\le 3 \\ \dfrac{1}{2(x-4)}+0.5, & 3<x\le 5,\ x\ne 4 \end{cases}$$



1. Over the interval [-1, 5], where is the function f discontinuous?

$x = 0, 1, 2, 4$

2. Complete the table for each value of x = c given.

X = c	$f(c)$	$\lim\limits_{x\to c^-} f(x)$	$\lim\limits_{x\to c^+} f(x)$	$\lim\limits_{x\to c} f(x)$	Does $f(c)=\lim\limits_{x\to c} f(x)$	Is $f(x)$ continuous at x = c?
0	0	−1	0	DNE	N	N
1	1	2	2	2	N	N
2	undef	0	0	0	N	N
3	0	0	0	0	Y	Y
4	undef	$-\infty$	∞	DNE	—	N

Using the table above, determine the three conditions that must exist in order for a function to be continuous:

a) The $\lim\limits_{x\to c} f(x)$ exists $\left(\lim\limits_{x\to c^-} f(x) = \lim\limits_{x\to c^+} f(x)\right)$

b) $f(c)$ is defined

c) $\lim\limits_{x\to c} f(x)$ equals $f(c)$

EXAMPLE: Let f be the function defined by $f(x)=\begin{cases} x+1, & x<2 \\ k(x-5)^2, & x\ge 2 \end{cases}$

where k stands for a constant.

1. Graph this function for $k=1$.

2. Use the definition of continuity to explain formally what must be true in order for f to be continuous at $x=2$.

$\lim\limits_{x\to 2^+} f(x) = \lim\limits_{x\to 2^-} f(x)$

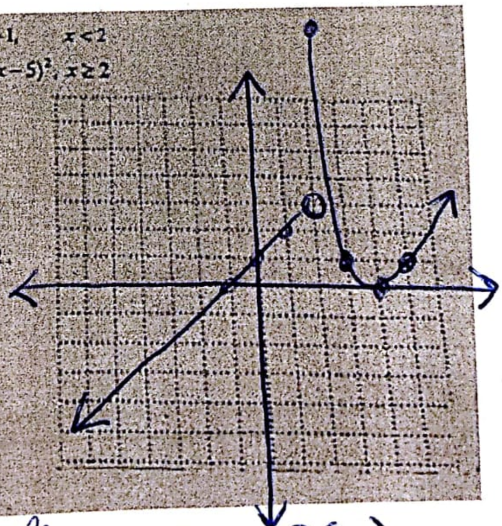
3. Find the value of k that makes f continuous at $x=2$. Then sketch f for this value of k.

$x+1 = k(x-5)^2$

$2+1 = k(2-5)^2$

$3 = k(-3)^2$

$k = \frac{1}{3}$



$\lim\limits_{x\to 2} f(x) = f(2)$

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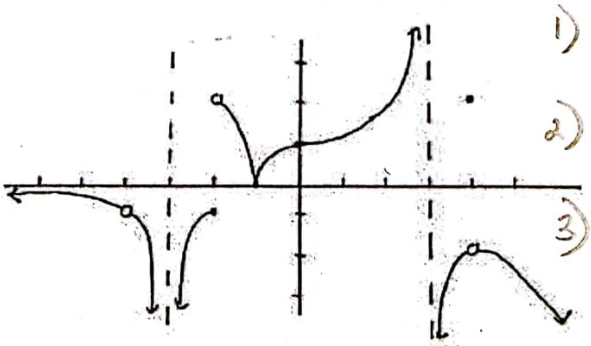
6. Use the graph below.
   a) Find all x values where the graph is discontinuous.
   b) For each discontinuous x value, name the type of discontinuity.
   c) For each discontinuous x value, state why it fails the formal definition of continuity.



1) $x = -4$ removable
$f(-4)$ is undefined
2) $x = -3$ infinite
$f(-3)$ is undefined
3) $x = -2$ jump/break
(non-removable)
$\lim_{x \to -2} f(x)$ DNE
4) $x = 3$ infinite
$\lim_{x \to 3} f(x)$ DNE
5) $x = 4$ removable $\lim_{x \to 4} f(x) \neq f(4)$

7. $f(x) = \begin{cases} \dfrac{x^2+5x+6}{x^2-4}, & \text{if } x \neq \pm 2 \\ -\dfrac{1}{4}, & \text{if } x = \pm 2 \end{cases}$ $\qquad \dfrac{(x+3)\cancel{(x+2)}}{(x-2)\cancel{(x+2)}}$

Where is $f(x)$ discontinuous? (A) Nowhere (B) $x = 2$ (C) $x = -2$ (D) $x = \pm 2$ (E) $\mathbb{R}$

Removable at $x = -2$ Infinite at $x = 2$

Find the value of a and b so that $f(x)$ is continuous

8. $f(x) = \begin{cases} ax - 1, & x < -1 \\ -x^2 + 1, & -1 \le x < 2 \\ \frac{1}{2}x + b, & x \ge 2 \end{cases}$

$ax - 1 = -x^2 + 1$
$-1a - 1 = -(-1)^2 + 1$
$-1a - 1 = 0$
$-1a = 1$
$\boxed{a = -1}$

$-x^2 + 1 = \frac{1}{2}x + b$
$-(2)^2 + 1 = \frac{1}{2}(2) + b$
$-3 = 1 + b$
$\boxed{b = -4}$

9. $f(x) = \begin{cases} 2x + a, & x \le -1 \\ x^2 + 1, & -1 < x \le 2 \\ bx - 1, & x > 2 \end{cases}$

$2x + a = x^2 + 1$
$2(-1) + a = (-1)^2 + 1$
$-2 + a = 2$
$\boxed{a = 4}$

$x^2 + 1 = bx - 1$
$4 + 1 = b(2) - 1$
$5 = 2b - 1$
$\boxed{b = 3}$

10. $f(x) = \begin{cases} \frac{x^2+7x+10}{x+2} & \text{if } x \neq -2 \\ b & \text{if } x = -2 \end{cases}$

$\dfrac{\cancel{(x+2)}(x+5)}{\cancel{(x+2)}} = x + 5$

$x + 5 = b$
$-2 + 5 = b$
$\boxed{b = 3}$

Are the following continuous at $x = 2$? If not, give the official reason according to the formal definition of continuity.

11. $f(x) = \begin{cases} x^2 & \text{if } x \le 2 \\ 3x - 2 & \text{if } x > 2 \end{cases}$ 4 4

Yes

12. $f(x) = \begin{cases} x^2 & \text{if } x < 2 \\ 3x - 2 & \text{if } x > 2 \end{cases}$ 4 4

No, $f(2)$ is undefined

13. $f(x) = \begin{cases} x^2 & \text{if } x \le 2 \\ 3x + 2 & \text{if } x > 2 \end{cases}$ 4 8

No
$\lim_{x \to 2} f(x)$ DNE

14. $f(x) = \begin{cases} x^2 & \text{if } x < 2 \\ 5 & \text{if } x = 2 \\ 3x - 2 & \text{if } x > 2 \end{cases}$ 4 5 4

No,
$\lim_{x \to 2} f(x) \neq f(2)$