CALCULUS BC
WORKSHEET ON SERIES

Work the following on notebook paper. Use your calculator only on 10(b).

1. Which of the following is a term in the Taylor series about $x = 0$ for the function $f(x) = \cos(2x)$?

(A) $-\frac{1}{2}x^2$ (B) $-\frac{4}{3}x^3$ (C) $\frac{2}{3}x^4$ (D) $\frac{1}{60}x^5$ (E) $\frac{4}{45}x^6$

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2. Find the values of x for which the series $\sum_{n=1}^{\infty} \frac{(x-2)^n}{n(-3)^n}$ converges.

(A) $x = 2$ (B) $-1 \le x < 5$ (C) $-1 < x \le 5$ (D) $-1 < x < 5$ (E) All real numbers

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3. Let $f(x) = \sum_{n=1}^{\infty} (\cos x)^{3n}$. Evaluate $f\left(\frac{2\pi}{3}\right)$.

(A) $-\frac{1}{7}$ (B) $-\frac{1}{9}$ (C) $\frac{1}{7}$ (D) $\frac{8}{9}$ (E) The series diverges.

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4. Find the sum of the geometric series $\frac{9}{8} - \frac{3}{4} + \frac{1}{2} - \frac{1}{3} + \ldots$

(A) $\frac{3}{5}$ (B) $\frac{5}{8}$ (C) $\frac{13}{24}$ (D) $\frac{27}{8}$ (E) $\frac{27}{40}$

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5. The series $x + x^3 + \frac{x^5}{2!} + \frac{x^7}{3!} + \ldots + \frac{x^{2n+1}}{n!} + \ldots$ is the Maclaurin series for

(A) $x\ln(1+x^2)$ (B) $x\ln(1-x^2)$ (C) e^{x^2} (D) xe^{x^2} (E) $x^2e^{x^2}$

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6. The coefficient of x^3 in the Taylor series for e^{2x} at $x = 0$ is

(A) $\frac{1}{6}$ (B) $\frac{1}{3}$ (C) $\frac{2}{3}$ (D) $\frac{4}{3}$ (E) $\frac{8}{3}$

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7. The Taylor polynomial of order 3 at $x = 0$ for $f(x) = \sqrt{1+x}$ is

(A) $1 + \frac{x}{2} - \frac{x^2}{4} + \frac{3x^3}{8}$ (B) $1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16}$ (C) $1 - \frac{x}{2} + \frac{x^2}{8} - \frac{x^3}{16}$

(D) $1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{8}$ (E) $1 - \frac{x}{2} + \frac{x^2}{4} - \frac{3x^3}{8}$

TURN->>>

8. The function f has a Taylor series about $x = 2$ that converges to $f(x)$ for all x in the interval of convergence. The nth derivative of f at $x = 2$ is given by $f^{(n)}(2) = \dfrac{(n+1)!}{3^n}$ for $n \geq 1$, and $f(2) = 1$.

(a) Write the first four terms and the general term of the Taylor series for f about $x = 2$.

(b) Find the radius of convergence for the Taylor series for f about $x = 2$. Show the work that leads to your answer.

(c) Let g be a function satisfying $g(2) = 3$ and $g'(x) = f(x)$ for all x. Write the first four terms and the general term of the Taylor series for g about $x = 2$.

(d) Does the Taylor series for g as defined in part (c) converge at $x = -2$? Give a reason for your answer.

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9. The function f is defined by the power series

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \ldots + \frac{(-1)^n x^{2n}}{(2n+1)!} + \ldots$$

for all real numbers x.

(a) Find $f'(0)$ and $f''(0)$. Determine whether f has a local maximum, a local minimum, or neither at $x = 0$. Give a reason for your answer.

(b) Show that the second-degree Taylor polynomial for f about $x = 0$ approximates $f(1)$ with error less than $\dfrac{1}{100}$.

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10. Let f be a function that has derivatives of all orders on the interval $(-1, 1)$. Assume that $f(0) = 6$, $f'(0) = 8$, $f''(0) = 30$, $f'''(0) = 48$, and $\left|f^{(n)}(x)\right| \leq 75$ for all x in (0, 1).

(a) Write a third-degree Taylor polynomial for f about $x = 0$.

(b) Use your answer to (a) to estimate the value of $f(0.2)$. What is the maximum possible error in making this estimate? Justify your answer.

8)

(a) $f(2) = 1;\ f'(2) = \frac{2!}{3};\ f''(2) = \frac{3!}{3^2};\ f'''(2) = \frac{4!}{3^3}$

$$f(x) = 1 + \frac{2}{3}(x-2) + \frac{3!}{2!3^2}(x-2)^2 + \frac{4!}{3!3^3}(x-2)^3 + \cdots + \frac{(n+1)!}{n!3^n}(x-2)^n + \cdots$$

$$= 1 + \frac{2}{3}(x-2) + \frac{3}{3^2}(x-2)^2 + \frac{4}{3^3}(x-2)^3 + \cdots + \frac{n+1}{3^n}(x-2)^n + \cdots$$

3 :
- 1 : coefficients $\frac{f^{(n)}(2)}{n!}$ in first four terms
- 1 : powers of $(x-2)$ in first four terms
- 1 : general term

(b) $$\lim_{n\to\infty}\left|\frac{\frac{n+2}{3^{n+1}}(x-2)^{n+1}}{\frac{n+1}{3^n}(x-2)^n}\right| = \lim_{n\to\infty}\frac{n+2}{n+1}\cdot\frac{1}{3}|x-2|$$

$= \frac{1}{3}|x-2| < 1$ when $|x-2| < 3$

The radius of convergence is 3.

3 :
- 1 : sets up ratio
- 1 : limit
- 1 : applies ratio test to conclude radius of convergence is 3

(c) $g(2) = 3;\ g'(2) = f(2);\ g''(2) = f'(2);\ g'''(2) = f''(2)$

$$g(x) = 3 + (x-2) + \frac{1}{3}(x-2)^2 + \frac{1}{3^2}(x-2)^3 + \cdots + \frac{1}{3^n}(x-2)^{n+1} + \cdots$$

2 :
- 1 : first four terms
- 1 : general term

(d) No, the Taylor series does not converge at $x = -2$ because the geometric series only converges on the interval $|x-2| < 3$.

1 : answer with reason

Answers to Worksheet on Series

1. C
2. C
3. B
4. E
5. D
6. D
7. B

9. (a) $f'(0) = 0$ and $f''(0) = -\frac{1}{3}$

Since $f'(0) = 0$ and $f''(0) = -\frac{1}{3}$, f has a local maximum at $x = 0$ by the Second Derivative Test.

(b) $f(1) = 1 - \frac{1}{3!} + \frac{1}{5!} - ...$

This is an alternating series whose terms are decreasing in size so the error involved in approximating $f(1)$ by this polynomial is less in magnitude than the first truncated term.

Therefore $|\text{Error}| < \frac{1}{5!} = \frac{1}{120} < \frac{1}{100}$ by the Alternating Series Remainder.

10. (a) $6 + 8x + 15x^2 + 8x^3$

(b) $f(0.2) \approx 8.264$

By the Lagrange Error Bound, $|R_3(x)| = \left|\frac{f^{(4)}(z)x^4}{4!}\right| \le \frac{75x^4}{4!}$

so $|\text{Error}| = |R_3(0.2)| \le \frac{75(0.2)^4}{4!} = 0.005$