9.8 Power Series

Warm-Up: Write 3rd degree Taylor poly with $c=8$

$f(x) = \sqrt[3]{x} = x^{1/3}$ $\quad f(8) = 2$

$f'(x) = \frac{1}{3}x^{-2/3}$ $\quad \frac{1}{12}$

$f''(x) = -\frac{2}{9}x^{-5/3}$ $\quad -\frac{1}{144}$

$f'''(x) = \frac{10}{27}x^{-8/3}$ $\quad \frac{5}{3456}$

$$P_3(x) = 2 + \frac{1}{12}(x-8) - \frac{1}{144 \cdot 2!}(x-8)^2 + \frac{5}{3456 \cdot 3!}(x-8)^3$$

$$P_3(x) = 2 + \frac{1}{12}(x-8) - \frac{1}{288}(x-8)^2 + \frac{5}{20736}(x-8)^3$$

Def of Power Series:

* centered at c

$$\sum_{n=0}^{\infty} a_n (x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + \ldots + a_n(x-c)^n$$

* special case at $c=0$

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots a_n x^n$$

* can be viewed as function of x

$$f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$$ (c is always in domain)

where domain of $f(x)$ is set of x values where series converges

Convergence of Power Series:

1) Converges at c $\qquad R=0$

2) Converges absolutely for all x $\qquad R=\infty$

3) Converges absolutely for $|x-c|<R$ and diverges for $|x-c|>R$

Do First ∇

* (R) Radius of convergence = distance from midpt of interval of convergence to endpt

Interval of Convergence = set of all values of x for which power series converges

(1) * $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n^2}$ Ratio Test: $\displaystyle\lim_{n\to\infty}\left|\frac{x^{n+1}}{(n+1)^2}\cdot\frac{n^2}{x^n}\right|$

$= \displaystyle\lim_{n\to\infty}\left|\frac{\cancel{x^n}\cdot x}{(n+1)^2}\cdot\frac{n^2}{\cancel{x^n}}\right| = \lim_{n\to\infty}\left|\frac{x\cdot n^2}{\underbrace{(n+1)^2}_{=1}}\right| = 1\cdot|x|$

Converges if <1

$|x|<1$

$-1<x<1$

*check endpts:

$x=-1$: $\dfrac{(-1)^n}{n^2}$ AST converges

$x=1$: $\dfrac{(1)^n}{n^2}$ p-series converges

IOC: $-1\le x\le 1$

ROC: 1

(2) * $\sum_{n=0}^{\infty} n\left(\frac{2x-5}{3}\right)^n$ Ratio Test: $\lim_{n\to\infty}\left|\frac{(n+1)\left(\frac{2x-5}{3}\right)^{n+1}}{n\left(\frac{2x-5}{3}\right)^n}\right|$

$= \lim_{n\to\infty}\left|\underbrace{\frac{(n+1)}{n}}_{=1}\cdot\left(\frac{2x-5}{3}\right)\right| = 1\cdot\left|\frac{2x-5}{3}\right| < 1$

$-1 < \frac{2x-5}{3} < 1$

$-3 < 2x-5 < 3$

$2 < 2x < 8$

$1 < x < 4$

* check endpts:

$x=1$: $n(-1)^n$ AST div

$x=4$: $n(1)^n$ div

IOC: $1 < x < 4$

ROC: $\frac{3}{2}$

(3) $\sum_{n=0}^{\infty} (2n)!\left(\frac{x}{2}\right)^n$

$\lim_{n\to\infty}\left|\frac{(2(n+1))!\left(\frac{x}{2}\right)^{n+1}}{(2n)!\left(\frac{x}{2}\right)^n}\right| = \lim_{n\to\infty}\left|\frac{(2n+2)(2n+1)\cancel{(2n)}\ldots\cancel{\left(\frac{x}{2}\right)^n}\left(\frac{x}{2}\right)^1}{\cancel{(2n)}(2n-1)\ldots\cancel{\left(\frac{x}{2}\right)^n}}\right|$

$= \lim_{n\to\infty}\left|(4n^2+6n+2)\left(\frac{x}{2}\right)\right| = \infty$ diverges

Converges only at $x=0$

↳ when $(x-c)^n = 0$

ROC: 0

IOC: $x=0$

④ $\sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$

$$\lim_{n\to\infty} \left| \frac{x^{2(n+1)+1}}{(2(n+1)+1)!} \cdot \frac{(2n+1)!}{x^{2n+1}} \right| = \lim_{n\to\infty} \left| \frac{x^{2n+3}}{(2n+3)!} \cdot \frac{(2n+1)!}{x^{2n+1}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{\cancel{x^{2n}} \cdot x^3}{(2n+3)(2n+2)\cancel{(2n+1)}\ldots} \cdot \frac{\cancel{(2n+1)(2n)\ldots}}{\cancel{x^{2n}} \cdot x} \right|$$

$$= \lim_{n\to\infty} \left| \frac{x^2}{(2n+3)(2n+2)} \right| = 0 \cdot |x^2| < 1$$

always converges

IOC: $(-\infty, \infty)$

ROC: ∞

⑤ $\sum_{n=1}^{\infty} \frac{(-1)^n (x+1)^n}{2^n}$ $\lim_{n\to\infty} \left| \frac{(-1)^{n+1}(x+1)^{n+1}}{2^{n+1}} \cdot \frac{2^n}{(-1)^n (x+1)^n} \right|$

$$= \lim_{n\to\infty} \left| \frac{\cancel{(-1)^n} \cdot (-1) \cancel{(x+1)^n} (x+1)}{\cancel{2^n} \cdot 2} \cdot \frac{\cancel{2^n}}{\cancel{(-1)^n}\cancel{(x+1)^n}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(-1)(x+1)}{2} \right| < 1$$

$-1 < \frac{-x-1}{2} < 1$

$-2 < -x-1 < 2$

$-1 < -x < 3$

$1 > x > -3$

$-3 < x < 1$

*check:

$x = -3$ $\frac{(-1)^n(-2)^n}{2^n} = \frac{-1(1)^n \cdot -1\cancel{(2)^n}}{\cancel{2^n}}$

$= 1^n \neq 0$ di

$x = 1$ $\frac{(-1)^n \cdot \cancel{2^n}}{\cancel{2^n}} =$ di

(6) $\sum_{n=1}^{\infty} \frac{x^n}{n}$ $\lim_{n\to\infty} \left| \frac{x^{n+1}}{n+1} \cdot \frac{n}{x^n} \right|$

$= \lim_{n\to\infty} \left| \frac{\cancel{x^n} \cdot x}{(n+1)} \cdot \frac{n}{\cancel{x^n}} \right| = \left| \underbrace{\frac{n}{n+1}}_{=1} \cdot x \right| = |x| < 1$

$-1 < x < 1$

* check:

$x = -1$ $\frac{(-1)^n}{n}$ AST: con

$x = 1$ $\frac{1^n}{n}$ p-series: di

IOC: $-1 < x < 1$

ROC: 1

(7) $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$

$\lim_{n\to\infty} \left| \frac{(-1)^{n+1} x^{2(n+1)+1}}{(2(n+1)+1)!} \cdot \frac{(2n+1)!}{(-1)^n x^{2n+1}} \right|$

$= \lim_{n\to\infty} \left| \frac{\cancel{(-1)^n} \cdot (-1)^1 \cancel{x^{2n}} \cdot x^3}{(2n+3)(2n+2)\cancel{(2n+1)} + \dots} \cdot \frac{\cancel{(2n+1)(2n)\dots}}{\cancel{(-1)^n} \cancel{x^{2n}} \cdot x} \right|$

$= \lim_{n\to\infty} \left| \frac{-x^2}{(2n+3)(2n+2)} \right| = 0 \cdot |-x^2| < 1$

always converges

IOC: $(-\infty, \infty)$

ROC: ∞