9.6 Warm-Up

1) $\sum_{n=1}^{\infty} \frac{3}{n^2+5}$ small Compare: $\frac{1}{n^2}$ converges big

$\therefore$ converges

2) $\sum_{n=1}^{\infty} \frac{1}{2^n+1}$ small compare: $\frac{1}{2^n} = \left(\frac{1}{2}\right)^n$ converges big

$\therefore$ converges

3) $\sum_{n=1}^{\infty} \frac{3n^2}{2n^2+1}$ nth term: $\lim_{n\to\infty} \frac{3n^2}{2n^2+1} = \frac{\infty}{\infty}$

L'H: $\frac{6n}{4n} = \frac{3}{2} \neq 0$ diverges

4) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n+4}$ $\left|\frac{(-1)^n}{n+4}\right| = \frac{1}{n+4}$ Compare to $\frac{1}{n}$ diverges

$\lim_{n\to\infty} \frac{1}{n+4} = 0$ $\therefore$ conditionally convergent

5) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}\cdot 4}{3n^2-1}$ $\left|\frac{(-1)^{n-1}\cdot 4}{3n^2-1}\right| = \frac{4}{3n^2-1}$ compare to $\frac{1}{n^2}$ con

$\therefore$ absolutely converges

9.6 Ratio and Root Tests

Ratio Test:

Let $\sum a_n$ be a series with nonzero terms

1) $\sum a_n$ converges absolutely if $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1$

2) $\sum a_n$ diverges if $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| > 1$ or $= \infty$

3) Inconclusive if $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$

① $\displaystyle\sum_{n=0}^{\infty} \frac{2^n}{n!}$ $\quad \displaystyle\lim_{n\to\infty} \frac{\frac{2^{n+1}}{(n+1)!}}{\frac{2^n}{n!}} = \frac{2^{n+1}}{(n+1)!}\cdot\frac{n!}{2^n}$

$$\lim_{n\to\infty} \frac{\cancel{2^n}\cdot 2^1}{(n+1)\cancel{(n)}\cancel{(n-1)}\ldots}\cdot\frac{\cancel{n(n-1)\ldots}}{\cancel{2^n}} = \frac{2}{n+1} = 0$$

$\therefore$ converges absolutely

② $\displaystyle\sum_{n=0}^{\infty} \frac{n^2\, 2^{n+1}}{3^n}$ $\quad \displaystyle\lim_{n\to\infty} \frac{\frac{(n+1)^2\cdot 2^{(n+1)+1}}{3^{n+1}}}{\frac{n^2\, 2^{n+1}}{3^n}}$

$$= \frac{(n+1)^2\cdot \cancel{2^n}\cdot \cancel{2^2}^{\,2}}{\cancel{3^n}\cdot 3}\cdot\frac{\cancel{3^n}}{n^2\cdot\cancel{2^n}\cdot\cancel{2}} = \frac{2(n+1)^2}{3n^2} = \frac{2}{3}$$

$\therefore$ converges absolutely

(3) $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n+1}$ $\quad \displaystyle\lim_{n\to\infty} \frac{\dfrac{\sqrt{n+1}}{(n+1)+1}}{\dfrac{\sqrt{n}}{n+1}}$

$$= \frac{\sqrt{n+1}}{n+2}\cdot\frac{n+1}{\sqrt{n}} = 1$$ ∴ inconclusive

(4) $\displaystyle\sum_{n=0}^{\infty} \frac{3^n}{n!}$ $\quad \displaystyle\lim_{n\to\infty} \frac{\dfrac{3^{n+1}}{(n+1)!}}{\dfrac{3^n}{n!}}$

$$= \frac{3^{\cancel{n}}\cdot 3}{(n+1)\cancel{(n)}\cancel{(n-1)}\ldots}\cdot\frac{\cancel{n}\cancel{(n-1)}\ldots}{\cancel{3^n}} = \frac{3}{n+1} = 0$$ ∴ converges absolutely

Root Test:

Let $\sum a_n$ be a series

1. $\sum a_n$ converges absolutely if $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} < 1$
2. $\sum a_n$ diverges if $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} > 1$ or $= \infty$
3. Inconclusive if $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} = 1$

① $\displaystyle\sum_{n=1}^{\infty} \frac{e^{2n}}{n^n} = \left(\frac{e^2}{n}\right)^n$

$\lim\limits_{n\to\infty} \sqrt[n]{\left(\frac{e^2}{n}\right)^n} = \left|\frac{e^2}{n}\right| = 0$ $\therefore$ converges absolutely

② X $\displaystyle\sum_{n=1}^{\infty} \left(\frac{-3n}{2n+1}\right)^{3n}$

$\lim\limits_{n\to\infty} \sqrt[n]{\left(\frac{-3n}{2n+1}\right)^{3n}} = \left|\frac{-3n}{2n+1}\right|^3 = \frac{3}{2}$ $\therefore$ diverges

③ $\displaystyle\sum_{n=1}^{\infty} \left(\frac{\ln n}{n}\right)^n$ $\lim\limits_{n\to\infty} \sqrt[n]{\left(\frac{\ln n}{n}\right)^n} = \frac{\ln n}{n} \overset{\text{L'H:}}{=} \frac{\frac{1}{n}}{1} = 0$

$\therefore$ converges absolutely

④ $\displaystyle\sum_{n=1}^{\infty} \left(\frac{2n}{n+1}\right)^n$ $\lim\limits_{n\to\infty} \sqrt[n]{\left(\frac{2n}{n+1}\right)^n}$

$= \frac{2n}{n+1} = 2$ $\therefore$ diverges