9.4 Comparison of Series

Direct Comparison Test:

Let $0 < a_n \le b_n$ for all n. Then,

- If $\sum_{n=1}^{\infty} b_n$ converges, $\sum_{n=1}^{\infty} a_n$ converges
- If $\sum_{n=1}^{\infty} a_n$ diverges, $\sum_{n=1}^{\infty} b_n$ diverges

* terms of series must be positive

★ If "larger" series converges, then "smaller" series converges

★ If "smaller" series diverges, then "larger" series diverges

*1) $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ Resembles: $\frac{1}{n^2}$

SMALL BIG

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ Converges so $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ converges

*2) $\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n^2}-2}$ Resembles $\frac{1}{n^{2/3}}$

BIG SMALL

$\sum_{n=1}^{\infty} \frac{1}{n^{2/3}}$ diverges so $\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n^2}-2}$ diverges

$*3)$ $\displaystyle\sum_{n=1}^{\infty} \underset{\text{SMALL}}{\frac{\cos^2 n}{n^3+1}}$ Resembles $\underset{\text{BIG}}{\dfrac{1}{n^3}}$

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges, both converge!

4) $\displaystyle\sum_{n=1}^{\infty} \underset{\text{small}}{\frac{1}{n^4+1}}$ Resembles $\underset{\text{big}}{\dfrac{1}{n^4}}$

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^4}$ converges, both converge!

5) $\displaystyle\sum_{n=1}^{\infty} \underset{\text{big}}{\frac{1}{\sqrt{3n-1}}}$ Resembles $\underset{\text{small}}{\dfrac{1}{\sqrt{3n}}} = \dfrac{1}{\sqrt{3}\,n^{1/2}}$

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{3}\,n^{1/2}}$ diverges, both diverge!

6) $\displaystyle\sum_{n=1}^{\infty} \underset{\text{small}}{\frac{3^n}{4^n+5}}$ Resembles $\underset{\text{big}}{\left(\frac{3}{4}\right)^n}$

$\displaystyle\sum_{n=1}^{\infty} \left(\frac{3}{4}\right)^n$ converges, both converge!

$*7)$ $\displaystyle\sum_{n=1}^{\infty} \underset{\text{small}}{\frac{1}{3+\sqrt{n}}}$ Resembles $\dfrac{1}{\sqrt{n}} = \underset{\text{big}}{\dfrac{1}{n^{1/2}}}$ big diverges $\rightarrow$ fails

Try: $\underset{\text{small}}{\dfrac{1}{n}}$ $\underset{\text{big}}{\dfrac{1}{3+\sqrt{n}}}$ $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^1}$ small diverges, both <u>diverge</u>!

Limit Comparison Test:

If $a_n > 0$ and $b_n > 0$ for all $n \geq N$

(N is positive integer)

* • If $\lim_{n\to\infty} \frac{a_n}{b_n} = c$, $0 < c < \infty$, then both $\sum a_n$ and $\sum b_n$ converge or both diverge

• If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$, then $\sum a_n$ converges if $\sum b_n$ converges

• If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$, then $\sum a_n$ diverges if $\sum b_n$ diverges

*1) $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+1}$

Comparison: $\frac{n^{1/2}}{n^2} = n^{-3/2} = \frac{1}{n^{3/2}}$ converges

$$\lim_{n\to\infty} \frac{\frac{\sqrt{n}}{n^2+1}}{\frac{1}{n^{3/2}}} = \frac{\sqrt{n}}{n^2+1} \cdot n^{3/2} = \frac{n^2}{n^2+1} = 1$$

$\therefore$ series converges

*2) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$

Comparison:
$\frac{1}{2^n} = \left(\frac{1}{2}\right)^n$ converges

$$\lim_{n\to\infty} \frac{\frac{1}{2^n-1}}{\left(\frac{1}{2}\right)^n} = \frac{1}{2^n-1} \cdot 2^n = 1$$

$\therefore$ series converges

*3) $\displaystyle\sum_{n=1}^{\infty} \frac{3n-1}{n^2+1}$

Comparison:
$\frac{n}{n^2} = \frac{1}{n}$ diverges

$$\lim_{n\to\infty} \frac{3n-1}{n^2+1} \cdot \frac{n}{1} = \frac{3n^2-n}{n^2+1} = 3$$ $\therefore$ diverges

4) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n+1}$

Comparison:
$\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n} = \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{n}$ diverges

$$\lim_{n\to\infty} \frac{1}{2n+1} \cdot 2n = 1$$ $\therefore$ diverges

5) $\displaystyle\sum_{n=1}^{\infty} \frac{3}{\sqrt{n^2-4}}$

Comparison:
$\frac{1}{n}$ diverges

$$\lim_{n\to\infty} \frac{3}{\sqrt{n^2-4}} \cdot n = 3$$ $\therefore$ diverges

6) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2-10}{4n^5+n^3}$

Comparison:
$\frac{1}{n^3}$ converges

$$\lim_{n\to\infty} \frac{n^2-10}{4n^5+n^3} \cdot n^3 = \frac{n^5-10n^3}{4n^5+n^3} = \frac{1}{4}$$ $\therefore$ converges