Sum of Infinite Geometric Series WS :

1) $\sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n$

$r = \frac{2}{3}$

$\frac{1}{1-\frac{2}{3}} = \frac{1}{\frac{1}{3}} = \boxed{3}$

2) $\sum_{n=0}^{\infty} \left(\frac{-2}{\pi}\right)^n$

$r = \frac{-2}{\pi}$

$\frac{1}{1-\left(-\frac{2}{\pi}\right)} = \frac{1}{\frac{\pi+2}{\pi}}$

$= \boxed{\frac{\pi}{\pi+2}}$

3) $\sum_{n=0}^{\infty} \left(\frac{4}{3}\right)^n$

$r=\frac{4}{3}$ $\therefore$ diverges

4) $\frac{1}{3^3} + \frac{1}{3^4} + \frac{1}{3^5} \dots$

$r = \frac{1}{3}$

$\frac{\frac{1}{27}}{1-\frac{1}{3}} = \frac{1}{27} \cdot \frac{3}{2}$

$= \boxed{\frac{1}{18}}$

5) $\sum_{n=0}^{\infty} \left(\frac{e}{2}\right)^n$

$r = \frac{e}{2}$

$\therefore$ diverges

6) $\sum_{n=3}^{\infty} \frac{3^n}{11^n}$

$\sum_{n=3}^{\infty} \left(\frac{3}{11}\right)^n$

$r = \frac{3}{11}$ $\frac{\frac{27}{1331}}{1-\frac{3}{11}}$

$\frac{27}{1331} \cdot \frac{11}{8} = \boxed{\frac{27}{968}}$

7) $1 + \frac{2}{7} + \frac{2^2}{7^2} + \frac{2^3}{7^3} \dots$

$r = \frac{2}{7}$

$\frac{1}{1-\frac{2}{7}} = \boxed{\frac{7}{5}}$

8) $\sum_{n=2}^{\infty} e^{3-2n}$

$e^3 \cdot (e^{-2})^n$

$r = \frac{1}{e^2}$

$\frac{\frac{1}{e}}{1-\frac{1}{e^2}} = \frac{1}{e} \cdot \frac{e^2}{e^2-1}$

$= \boxed{\frac{e}{e^2-1}}$

9) $\sum_{n=0}^{\infty} \frac{93^n + 4^{n-2}}{5^n}$ $\quad 4^n \cdot 4^{-2}$

$\sum_{n=0}^{\infty} \left(\frac{93}{5}\right)^n + \sum_{n=0}^{\infty} \frac{1}{16}\left(\frac{4}{5}\right)^n$

$\downarrow$

diverges

10) $\frac{64}{49} + \frac{8}{7} + 1 + \frac{7}{8} + \dots$

$r = \frac{7}{8}$

$\frac{\frac{64}{49}}{1-\frac{7}{8}} = \frac{64}{49} \cdot 8$

$= \boxed{\frac{512}{49}}$

11) $\sum_{n=0}^{\infty} \frac{7 \cdot 3^n}{5^n}$

$7 \cdot \left(\frac{3}{5}\right)^n$

$r = \frac{3}{5}$

$\frac{7}{1-\frac{3}{5}} \quad 7 \cdot \frac{5}{2}$

$= \boxed{\frac{35}{2}}$

12) $\sum_{n=0}^{\infty} \frac{8+2^n}{5^n}$ $\quad \boxed{\frac{35}{3}}$

$\sum_{n=0}^{\infty} \frac{8}{5^n} + \sum_{n=0}^{\infty} \frac{2^n}{5^n}$

$8 \cdot \left(\frac{1}{5}\right)^n \quad \left(\frac{2}{5}\right)^n$

$\frac{8}{1-\frac{1}{5}} = 8 \cdot \frac{5}{4} = 10 \quad \frac{1}{1-\frac{2}{5}} = \frac{5}{3}$

13) $\sum_{n=3}^{\infty} 2\left(-\frac{3}{4}\right)^n$

$r = -\frac{3}{4}$

$\frac{-\frac{27}{32}}{1-\left(-\frac{3}{4}\right)} = -\frac{27}{56}$

14) $\sum_{0}^{\infty} e^{-n}$

$r = \frac{1}{e}$ $\left(\frac{1}{e}\right)^n$

$\frac{1}{1-\frac{1}{e}} = \frac{1}{\frac{e-1}{e}}$

$= \frac{e}{e-1}$

15) B + C

16) B + C

Calculus BC – Sum of an infinite Geometric Series

Answers.

1. 3
2. $\dfrac{\pi}{\pi+2}$
3. Diverges
4. 1/18
5. Diverges
6. 27/968
7. 7/5
8. $\dfrac{e}{e^2-1}$
9. Diverges
10. 512/49
11. 35/2
12. ~~25/3~~ 35/3
13. -27/56
14. $\dfrac{e}{e-1}$
15. B & C
16. B & C

9.2 Telescoping Series, n^{th} Term Test, Integral Test, P-Series

① Telescoping Series: $\sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+1}$

* rewrite w/ partial fractions
* cancel terms

1) $$\sum_{n=1}^{\infty} \frac{8}{(n+1)(n+2)} = 8\sum_{n=1}^{\infty}\left(\frac{1}{n+1} - \frac{1}{n+2}\right)$$

$$= 8\left[\underbrace{\left(\frac{1}{2}-\cancel{\frac{1}{3}}\right)}_{n=1}+\underbrace{\left(\cancel{\frac{1}{3}}-\cancel{\frac{1}{4}}\right)}_{n=2}+\underbrace{\left(\cancel{\frac{1}{4}}-\frac{1}{5}\right)}_{n=3} \cdots\right.$$

$$= 8\left(\frac{1}{2}\right) = \boxed{4}$$

$$\frac{1}{(n+1)(n+2)} = \frac{A}{n+1} + \frac{B}{n+2}$$

$$1 = A(n+2) + B(n+1)$$

$n=-2$: $1 = B(-1)$ $B=-1$

$n=-1$: $1 = A(1)$ $A=1$

2) $$\sum_{n=2}^{\infty} \frac{1}{n^2-1}$$

$$\frac{1}{(n+1)(n-1)} = \frac{A}{n+1} + \frac{B}{n-1}$$

$$1 = A(n-1) + B(n+1)$$

$n=1$: $1 = B(2)$ $B=\frac{1}{2}$

$n=-1$: $1 = A(-2)$ $A=-\frac{1}{2}$

$$\sum_{n=2}^{\infty}\left(\frac{\frac{1}{2}}{n-1} - \frac{\frac{1}{2}}{n+1}\right) = \frac{1}{2}\sum_{n=2}^{\infty}\left(\frac{1}{n-1} - \frac{1}{n+1}\right)$$

$$= \frac{1}{2}\left[\left(1-\cancel{\frac{1}{3}}\right)+\left(\frac{1}{2}-\cancel{\frac{1}{4}}\right)+\left(\cancel{\frac{1}{3}}-\frac{1}{5}\right)+\left(\cancel{\frac{1}{4}}-\frac{1}{6}\right) \cdots\right.$$

$$= \frac{1}{2}\left(1+\frac{1}{2}\right) = \boxed{\frac{3}{4}}$$

② n^{th} Term Test
(ONLY tests for divergence!)

* If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ diverges

* If fails test, ask:
  Is it geometric? telescopic?

1) $\sum_{n=1}^{\infty} n^2$ $\quad \lim_{n\to\infty} n^2 = \infty \neq 0$ $\quad \therefore$ diverges

2) $\sum_{n=1}^{\infty} \frac{n+1}{n}$ $\quad \lim_{n\to\infty} \frac{n+1}{n} = 1 \neq 0$ $\quad \therefore$ diverges

3) $\sum_{n=1}^{\infty} \frac{n!}{2n!+1}$ $\quad \lim_{n\to\infty} \frac{n!}{2n!+1} = \frac{1}{2} \neq 0$ $\therefore$ diverges

$\rightsquigarrow$ mixed prac

3) Integral Test

If a_n is a positive sequence and $a_n = f(n)$ where $f(n)$ is a continuous, positive decreasing function, then:

$$\sum_{n=N}^{\infty} a_n \text{ and } \int_N^{\infty} f(x)\,dx \quad \text{both converge or both diverge}$$

1) Does $\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}}$ converge?

$$\int_1^{\infty} \frac{1}{x\sqrt{x}}\,dx = \lim_{b\to\infty} \int_1^b x^{-3/2}\,dx = \lim_{b\to\infty} -2x^{-1/2}\Big|_1^b$$

$$= \lim_{b\to\infty}\left(\frac{-2}{\sqrt{b}} + 2\right) = 2$$

Integral converges ∴ series converges (BUT not necessarily to 2)

2) C or D?

$$\sum_{n=2}^{\infty} \frac{\ln n}{n^3} \qquad \int_2^{\infty} \ln x \cdot x^{-3}\,dx \qquad \begin{array}{ll} u = \ln x & dv = x^{-3}dx \\ du = \frac{1}{x}dx & v = \frac{1}{-2x^2} \end{array}$$

$$\ln x \cdot -\frac{1}{2x^2} + \frac{1}{2}\int_2^b \frac{1}{x^2}\cdot\frac{1}{x}\,dx = \ln x \cdot -\frac{1}{2x^2} + \frac{1}{2}\int_2^b x^{-3}\,dx$$

$$-\frac{\ln x}{2x^2} + \frac{1}{2}\cdot\frac{1}{-2x^2}\Big|_2^b = \left(\underbrace{\cancel{\frac{-\ln b}{2b^2}}}_{0} - \underbrace{\cancel{\frac{1}{4b^2}}}_{0}\right) - \left(\frac{-\ln 2}{8} - \frac{1}{16}\right)$$

$$= \frac{\ln 2}{8} - \frac{1}{16} \qquad \therefore \text{ converges}$$

Mixed Practice: C or D?

1) $\displaystyle\sum_{n=1}^{\infty} \frac{n+1}{2n-1}$ $\qquad \displaystyle\lim_{n\to\infty} \frac{n+1}{2n} = \frac{1}{2} \neq 0 \therefore$ diverges

2) $\displaystyle\sum_{n=1}^{\infty} \frac{2}{4n^2-1}$

$$\frac{1}{(2n+1)(2n-1)} = \frac{A}{2n+1} + \frac{B}{2n-1}$$

$$1 = A(2n-1) + B(2n+1)$$

$$n = \tfrac{1}{2}: \quad 1 = B(2) \quad B = \tfrac{1}{2}$$

$$n = \tfrac{-1}{2}: \quad 1 = A(-2) \quad A = -\tfrac{1}{2}$$

$$= 2\sum_{n=1}^{\infty} \frac{1}{(2n+1)(2n-1)}$$

$$= 2 \cdot \frac{1}{2} \sum_{n=1}^{\infty} \left(\frac{1}{2n-1} - \frac{1}{2n+1}\right) = \left(1-\tfrac{1}{3}\right) + \left(\tfrac{1}{3}-\tfrac{1}{5}\right) + \left(\tfrac{1}{5}-\tfrac{1}{7}\right)\ldots$$

$$= \boxed{1}$$

3) $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{n^3}$ $\qquad \displaystyle\lim_{n\to\infty} \frac{3^n}{n^3} = \infty \neq 0 \quad \therefore$ diverges

4) $\displaystyle\sum_{n=0}^{\infty} \frac{1}{4^n}$ $\qquad \displaystyle\lim_{n\to\infty} \frac{1}{4^n} = 0$ $\quad$ FAILS!

Geometric: $1 \cdot \left(\frac{1}{4}\right)^n \quad \frac{a}{1-r} = \frac{1}{1-\frac{1}{4}} = \boxed{\frac{4}{3}}$

5) $\displaystyle\sum_{n=1}^{\infty} \ln\left(\frac{1}{n}\right)$ $\qquad \displaystyle\lim_{n\to\infty} \ln\left(\frac{1}{n}\right) = -\infty \neq 0$ $\therefore$ diverges

6) $\displaystyle\sum_{n=0}^{\infty} e^{-n}$ $\qquad \displaystyle\lim_{n\to\infty} \frac{1}{e^n} = 0$ $\quad$ FAILS!

Geometric: $1 \cdot \left(\frac{1}{e}\right)^n \quad \frac{a}{1-r} = \boxed{\frac{1}{1-\frac{1}{e}}}$

④ p - series Test

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \ldots$$

converges if $p > 1$
diverges if $0 < p \leq 1$

1) Harmonic Series

$$\sum_{n=1}^{\infty} \frac{1}{n} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$$

$p = 1$
diverges

C or D?

2) $\sum_{n=1}^{\infty} \frac{1}{n^{2/3}}$ $p < 1$ $\therefore$ diverges

3) $\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}}$ $p > 1$ $\therefore$ converges

4) $\sum_{n=1}^{\infty} \frac{1}{\sqrt[11]{n}}$ $p < 1$ $\therefore$ diverges

5) $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \ldots$

$\frac{1}{n^2}$ $p = 2$
$p > 1$
$\therefore$ converges

6) Find values of p for which series converges

$$\sum \frac{n^{(3p+1)}}{n^2}$$

$$\frac{1}{n^2 \cdot n^{-(3p+1)}}$$

$2 - (3p+1) > 1$
$-2 \qquad\qquad -2$
$-3p - 1 > -1$
$-3p > 0$
$p < 0$

3) $$\sum_{n=1}^{\infty} n \cdot e^{-n/2} = \sum_{n=1}^{\infty} \frac{n}{e^{n/2}}$$

$$\lim_{b\to\infty} \int_1^b x \cdot e^{-x/2}$$

$$\begin{array}{cc} u & dv \\ \hline +x & e^{-x/2} \\ -1 & -2e^{-x/2} \\ 0 & 4e^{-x/2} \end{array}$$

$$-2xe^{-x/2} - 4e^{-x/2} \Big|_1^b$$

$$\lim_{b\to\infty} \left[\left(\underset{0}{\cancel{\frac{-2b}{e^{b/2}}}} - \underset{0}{\cancel{\frac{4}{e^{b/2}}}} \right) - \left(\frac{-2}{e^{1/2}} - \frac{4}{e^{1/2}} \right) \right]$$

$$\frac{2}{e^{1/2}} + \frac{4}{e^{1/2}}$$ $\therefore$ converges

4) $$\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$$

$$\lim_{b\to\infty} \int_2^b \frac{1}{x\sqrt{\ln x}}\,dx$$

$u = \ln x$
$du = \frac{1}{x}\,dx$

$$\int u^{-1/2}\,du$$
$$= 2u^{1/2} = 2(\ln x)^{1/2}$$
$$= 2\sqrt{\ln x}$$

$$\lim_{b\to\infty} \left(2\sqrt{\ln x} \Big|_2^b \right)$$

$$= \lim_{b\to\infty} \left(2\sqrt{\ln b} - 2\sqrt{\ln 2} \right) = \infty$$

$\therefore$ diverges