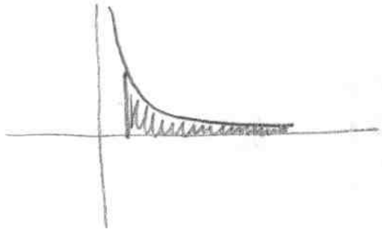
8.8 Improper Integrals Day 1

Intro: Find area under $f(x) = \frac{1}{x^2}$ on the interval $[1, \infty)$



Why can't we use FTC?

- Find area under $f(x)$ on $[1, t]$ where $t > 1$ and t is finite

$$A = \int_1^t \frac{1}{x^2}\,dx = -\frac{1}{x}\Big|_1^t = \frac{-1}{t} + 1$$

$$A = \lim_{t \to \infty} \left(\frac{-1}{t} + 1\right) = 1$$

IMPROPER INTEGRALS:

- an integral discontinuous on its interval
- an integral with infinite limits of integration

Rules:

1) If $f(x)$ is continuous on $[a,\infty)$, then
$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty}\int_a^b f(x)\,dx$$

2) If $f(x)$ is continuous on $(-\infty,b]$, then
$$\int_{-\infty}^b f(x)\,dx = \lim_{a\to-\infty}\int_a^b f(x)\,dx$$

3) If $f(x)$ is continuous on $(-\infty,\infty)$, then
$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^c f(x)\,dx + \int_c^\infty f(x)\,dx$$
(where c is any real #)

* Convergent - limit exists and is a finite #

* Divergent - limit either DNE or is $\pm\infty$

* In Rule 3, BOTH integrals must converge for integral to be convergent. Otherwise, integral diverges.

1. $\int_1^\infty \frac{1}{x}\,dx$

$= \lim_{b\to\infty} \int_1^b \frac{1}{x}\,dx$

$= \lim_{b\to\infty} \ln|x| \Big|_1^b$

$= \lim_{b\to\infty} (\ln|b| - \cancel{\ln 1})$ 0

$= \infty$

$\therefore$ diverges

2. $\int_1^\infty \frac{2}{x^3}\,dx$

$= \lim_{b\to\infty} \int_1^b 2x^{-3}\,dx$

$= \lim_{b\to\infty} \frac{2x^{-2}}{-2} \Big|_1^b$

$= \lim_{b\to\infty} -\frac{1}{x^2} \Big|_1^b$

$= \lim_{b\to\infty} \left(-\frac{1}{b^2} - \left(-\frac{1}{1}\right)\right)$

$= \lim_{b\to\infty} \left(-\frac{1}{b^2} + 1\right)$

$= \boxed{1}$ $\therefore$ converges

* Rule: If $a > 0$, then

$\int_a^\infty \frac{1}{x^p}\,dx$ is convergent if $p > 1$
divergent if $p \le 1$

3. $\int_0^\infty e^{-x}\,dx = \lim_{b\to\infty} \int_0^b e^{-x}\,dx$

$= \lim_{b\to\infty} -e^{-x} \Big|_0^b = \lim_{b\to\infty} \left(-e^{-b} - (-e^0)\right)$

$= \lim_{b\to\infty} \left(\frac{-1}{e^b} + 1\right) = \boxed{1}$ $\therefore$ converges

(4) $$\int_{-\infty}^{0} x e^{-2x}\,dx$$

$$= \lim_{a\to-\infty} \int_{a}^{0} x e^{-2x}\,dx$$

u	dv
$+x$	e^{-2x}
-1	$-\frac{1}{2}e^{-2x}$
0	$\frac{1}{4}e^{-2x}$

$$= \lim_{a\to-\infty} \left(-\frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x}\Big|_a^0\right)$$

$$= \lim_{a\to-\infty} \left(\left(0-\frac{1}{4}\right) - \left(-\frac{1}{2}ae^{-2a} - \frac{1}{4}e^{-2a}\right)\right)$$

$$= \lim_{a\to-\infty} \left(-\frac{1}{4} + \frac{a}{\underbrace{2e^{2a}}_{\text{small}}} + \frac{1}{\underbrace{4e^{2a}}_{\text{small}}}\right)$$

$$= \lim_{a\to-\infty} \left(\frac{-1}{4} - \infty + \infty\right) \quad \therefore \text{ divergent}$$

(5) $$\int_{1}^{\infty} x\ln x\,dx = \lim_{b\to\infty} \int_{1}^{b} x\ln x\,dx$$

$u = \ln x \quad dv = x\,dx$

$du = \frac{1}{x}dx \quad v = \frac{x^2}{2}$

$$= \lim_{b\to\infty} \left(\ln x\cdot\frac{x^2}{2} - \frac{1}{2}\int_{1}^{b} x^2\cdot\frac{1}{x}\,dx\right)$$

$$= \lim_{b\to\infty} \left(\frac{x^2}{2}\ln x - \frac{1}{2}\cdot\frac{x^2}{2}\Big|_1^b\right)$$

$$= \lim_{b\to\infty} \left[\left(\frac{b^2}{2}\ln b - \frac{1}{4}\cdot b^2\right) - \left(\frac{1}{2}\ln 1 - \frac{1}{4}\right)\right]$$

$$= \lim_{b\to\infty} \left(\infty - \infty + \frac{1}{4}\right) \quad \therefore \text{ diverges}$$

⑥ $\displaystyle\int_{-\infty}^{\infty} e^{2x}\,dx$

$$= \int_{-\infty}^{0} e^{2x}\,dx + \int_{0}^{\infty} e^{2x}\,dx$$

$$= \lim_{a\to-\infty} \int_{a}^{0} e^{2x}\,dx + \lim_{b\to\infty} \int_{0}^{b} e^{2x}\,dx$$

$$= \lim_{a\to-\infty} \frac{1}{2}e^{2x}\Big|_{a}^{0} + \lim_{b\to\infty} \frac{1}{2}e^{2x}\Big|_{0}^{b}$$

$$= \lim_{a\to-\infty} \left(\frac{1}{2}e^{0} - \frac{1}{2}e^{2a}\right) + \lim_{b\to\infty} \left(\frac{1}{2}e^{2b} - \frac{1}{2}e^{0}\right)$$

$$= \lim_{a\to-\infty} \left(\frac{1}{2} - \frac{1}{2}e^{2a}\right) + \lim_{b\to\infty} \left(\frac{1}{2}e^{2b} - \frac{1}{2}\right)$$

$$\left(\frac{1}{2} - 0\right) + \left(\infty - \frac{1}{2}\right) \quad \therefore \text{ diverges}$$

⑦ $\displaystyle\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \int_{-\infty}^{0} \frac{1}{1+x^2}\,dx + \int_{0}^{\infty} \frac{1}{1+x^2}\,dx$

$$= \lim_{a\to-\infty} \int_{a}^{0} \frac{1}{1+x^2}\,dx + \lim_{b\to\infty} \int_{0}^{b} \frac{1}{1+x^2}\,dx$$

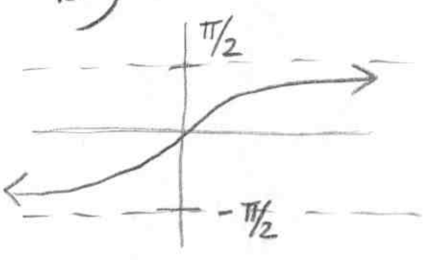
$$\lim_{a\to-\infty} \tan^{-1}x\Big|_{a}^{0} + \lim_{b\to\infty} \tan^{-1}x\Big|_{0}^{b}$$

$$\lim_{a\to-\infty} \left(\tan^{-1}0 - \tan^{-1}a\right) + \lim_{b\to\infty} \left(\tan^{-1}b - \tan^{-1}0\right)$$

$$\lim_{a\to-\infty} \left(-\tan^{-1}a\right) + \lim_{b\to\infty} \left(\tan^{-1}b\right)$$

$$-\left(-\frac{\pi}{2}\right) + \frac{\pi}{2}$$

$$= \boxed{\pi} \quad \therefore \text{ converges}$$



8 $\displaystyle\int_{-\infty}^{0} \frac{2}{x^2-4x+3}$

$\boxed{\ln 3}$

8.8 Day 2

Rules:

1) If $f(x)$ is continuous on $[a, b)$ and not continuous at $x=b$, then

$$\int_a^b f(x)\,dx = \lim_{t \to b^-} \int_a^t f(x)\,dx$$

2) If $f(x)$ is continuous on $(a, b]$ and not continuous at $x=a$, then

$$\int_a^b f(x)\,dx = \lim_{t \to a^+} \int_t^b$$

3) If $f(x)$ is not continuous at $x=c$ where $a<c<b$ and $\int_a^c f(x)\,dx$ and $\int_c^b f(x)\,dx$ are convergent, then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

① $\displaystyle\int_0^3 \frac{1}{\sqrt{3-x}}\,dx$ discontinuous at $x=3$

$$= \lim_{t\to 3^-} \int_0^t \frac{1}{\sqrt{3-x}}\,dx = \lim_{t\to 3^-}\left(\frac{(3-x)^{1/2}}{\frac{1}{2}\cdot -1}\Big|_0^t \right)$$

$(3-x)^{-1/2}$

$$= \lim_{t\to 3^-}\left(-2\sqrt{3-x}\,\Big|_0^t\right) = \lim_{t\to 3^-}\left(-2\sqrt{3-t} + 2\sqrt{3-0}\right)$$

$$= \lim_{t\to 3^-}\left(\underbrace{\cancel{-2\sqrt{3-3}}}_{0} + 2\sqrt{3}\right) = \boxed{2\sqrt{3}}$$

② $\displaystyle\int_{-2}^3 \frac{1}{x^3}\,dx$ discontinuous @ $x=0$

$$= \int_{-2}^0 \frac{1}{x^3}\,dx + \int_0^3 \frac{1}{x^3}\,dx$$

$\downarrow$

$$\int_{-2}^0 \frac{1}{x^3}\,dx = \lim_{t\to 0^-}\int_{-2}^t x^{-3}\,dx = \lim_{t\to 0^-}\left(\frac{-1}{2x^2}\Big|_{-2}^t\right)$$

$$= \lim_{t\to 0^-}\left(\frac{-1}{2t^2} - \left(\frac{-1}{2(-2)^2}\right)\right)$$

$$= -\infty + \frac{1}{8} \quad \therefore \text{ divergent}$$

③ $\displaystyle\int_0^1 \sqrt{\frac{1+x}{1-x}}\,dx$ discontinuous at $x=1$

$$= \lim_{t\to 1^-} \int_0^t \sqrt{\frac{1+x}{1-x}}\,dx \qquad \frac{\sqrt{1+x}}{\sqrt{1-x}} \cdot \frac{\sqrt{1+x}}{\sqrt{1+x}}$$

$$= \lim_{t\to 1^-} \int_0^t \frac{1+x}{\sqrt{1-x^2}}\,dx \qquad = \frac{1+x}{\sqrt{1-x^2}}$$

$$= \lim_{t\to 1^-} \left(\int \frac{1}{\sqrt{1-x^2}}\,dx + \int \frac{x}{\sqrt{1-x^2}}\,dx \right)$$

$u = 1-x^2$
$du = -2x\,dx$
$-\frac{1}{2}\,du = x\,dx$

$$= \lim_{t\to 1^-} \left(\sin^{-1}x \Big| - \frac{1}{2} \int u^{-1/2}\,du \right)$$

$$= \lim_{t\to 1^-} \left(\sin^{-1}x \Big| - \underbrace{\frac{1}{2} \cdot \frac{2}{1}\, u^{1/2}}_{-\sqrt{1-x^2}} \Big|_0^t \right)$$

$$= \lim_{t\to 1^-} \left(\sin^{-1}t - \sqrt{1-t^2} - \left(\sin^{-1}0 - \sqrt{1} \right) \right)$$

$$= \lim_{t\to 1^-} \left(\sin^{-1}1 - 0 - 0 + 1 \right)$$

$$= \boxed{\frac{\pi}{2} + 1}$$

④ $\displaystyle\int_0^1 \frac{1}{x}\,dx$ discont. @ $x=0$

$$\lim_{t\to 0^+} \int_t^1 \frac{1}{x}\,dx = \lim_{t\to 0^+} \ln|x| \Big|_t^1$$

$\therefore$ divergent

$$= \lim_{t\to 0^+} \left(\ln 1 - \ln t \right) = \ln \frac{1}{t} = \boxed{\infty}$$

(5) $\displaystyle\int_0^3 \frac{1}{(x-1)^{2/3}}\,dx$ discont @ $x=1$

$$= \lim_{t\to 1^-}\int_0^t (x-1)^{-2/3}\,dx + \lim_{t\to 1^+}\int_t^3 (x-1)^{-2/3}\,dx$$

$$= \lim_{t\to 1^-} 3(x-1)^{1/3}\Big|_0^t + \lim_{t\to 1^+} 3(x-1)^{1/3}\Big|_t^3$$

$$\lim_{t\to 1^-}\left[3(t-1)^{1/3} - 3(0-1)^{1/3}\right] + \lim_{t\to 1^+}\left[3(3-1)^{1/3} - 3(t-1)^{1/3}\right]$$

$$= (0+3) + (3\sqrt[3]{2} - 0)$$

$$\boxed{3 + 3\sqrt[3]{2}}$$

(6) $\displaystyle\int_0^1 \frac{dx}{\sqrt{1-x^2}}$ discont @ $x=1$

$$\lim_{t\to 1^-}\int_0^t \frac{1}{\sqrt{1-x^2}}\,dx = \lim_{t\to 1^-} \sin^{-1}x\Big|_0^t$$

$$= \lim_{t\to 1^-}\left(\sin^{-1}t - \cancel{\sin^{-1}0}\right)$$

$$= \lim_{t\to 1^-}\left(\sin^{-1}1\right)$$

$$= \boxed{\frac{\pi}{2}}$$

⑦ $\displaystyle\int_0^4 \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,dx$ discont @ $x=0$

$\displaystyle\lim_{t\to 0^+}\int_t^4 \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,dx$

$u = -\sqrt{x} = -(x)^{1/2}$

$du = -\frac{1}{2}x^{-1/2}\,dx$

$-2\,du = \frac{1}{\sqrt{x}}\,dx$

$\displaystyle\lim_{t\to 0^+} -2\int e^u\,du$

$\displaystyle\lim_{t\to 0^+} -2e^{-\sqrt{x}}\Big|_t^4 = \lim_{t\to 0^+}\left(-2e^{-\sqrt{4}} - (-2e^{-\sqrt{t}})\right)$

$= -2e^{-2} + 2 = 2 - 2e^{-2}$

⑧ $\displaystyle\int_0^2 \frac{dx}{(x-1)^4} = \lim_{t\to 1^-}\int_0^t \frac{1}{(x-1)^4}\,dx + \lim_{t\to 1^+}\int_t^2 \frac{1}{(x-1)^4}\,dx$

discont @ $x=1$

$\displaystyle\lim_{t\to 1^-} \frac{(x-1)^{-3}}{-3}\Big|_0^t + \lim_{t\to 1^+}\frac{(x-1)^{-3}}{-3}\Big|_t^2$

$\displaystyle\frac{(t-1)^{-3}}{-3} - \frac{(-1)^{-3}}{-3} + \frac{(1)^{-3}}{-3} - \frac{(t-1)^{-3}}{-3}$

$\displaystyle\underbrace{\frac{-1}{3(t-1)^3}}_{\text{small}} + \frac{1}{3(1)} - \frac{1}{3\cdot 1} + \underbrace{\frac{1}{3(t-1)^3}}_{\text{small}}$

$-\infty + \cancel{\frac{1}{3} - \frac{1}{3}} + \infty$

$\therefore$ diverges

(9) $\displaystyle\int_0^1 \frac{dx}{\sqrt[3]{x}}$ discont @ $x=0$

$$\lim_{t\to 0^+} \int_t^1 x^{1-1/3}\,dx = \frac{3}{2}x^{2/3}\Big|_t^1$$

$$\lim_{t\to 0^+}\left(\frac{3}{2} - \frac{3}{2}t^{2/3}\right)$$

$$= \boxed{\frac{3}{2}}$$