8.5 Partial Fractions

* method used for evaluating integrals with rational expressions

* Degree NUM < Degree DENOM (if no → use long division)

1) * $\displaystyle\int \frac{5x-3}{x^2-2x-3}\,dx$

$(x-3)(x+1)$ $(x-3)(x+1)$

$$\frac{5x-3}{(x-3)(x+1)} = \frac{A}{(x-3)} + \frac{B}{(x+1)}$$ multiply by LCD

$$5x-3 = A(x+1) + B(x-3)$$

$x=-1$: $-8 = A(0) + B(-4)$
$-8 = -4B$
$B = 2$

$x=3$: $12 = A(4) + B(0)$
$12 = 4A$
$A = 3$

$$= \int\left(\frac{3}{x-3} + \frac{2}{x+1}\right)dx$$

$$= 3\int \frac{1}{x-3}\,dx + 2\int \frac{1}{x+1}\,dx$$

$$= 3\ln|x-3| + 2\ln|x+1| + C$$

2) $*$

$$\int \frac{1}{x^2-5x+6}\,dx$$

$$= \int \frac{1}{(x-2)(x-3)}\,dx$$

$(x-2)(x-3) \qquad (x-2)(x-3) \qquad (x-2)(x-3)$

$$\frac{1}{(x-2)(x-3)} = \frac{A}{(x-2)} + \frac{B}{(x-3)}$$

$$1 = A(x-3) + B(x-2)$$

$x=2$: $\quad 1 = A(-1) + B(0)$

$$1 = -1A$$

$$A = -1$$

$x=3$: $\quad 1 = A(0) + B(1)$

$$1 = 0 + B$$

$$B = 1$$

$$\int \left(\frac{-1}{x-2} + \frac{1}{x-3} \right) dx$$

$$= -\ln|x-2| + \ln|x-3| + C$$

$$= \ln\left|\frac{x-3}{x-2}\right| + C$$

3) $\int \frac{7x+6}{x^2+5x+6}\,dx$

$\int \frac{7x+6}{(x+2)(x+3)}\,dx \qquad \frac{7x+6}{(x+2)(x+3)} = \frac{A}{x+2} + \frac{B}{x+3}$

$7x+6 = A(x+3) + B(x+2)$

$x=-3: \quad -15 = A(0) + B(-1)$
$-15 = -1B$
$B = 15$

$x=-2: \quad -8 = A(1) + B(0)$
$A = -8$

$\int \left(\frac{-8}{x+2} + \frac{15}{x+3}\right) dx = -8\ln|x+2| + 15\ln|x+3| + C$

4) * $\int \frac{5x+3}{x^3-2x^2-3x}\,dx \qquad \frac{5x+3}{x(x-3)(x+1)} = \frac{A}{x} + \frac{B}{(x-3)} + \frac{C}{(x+1)}$

$x(x^2-2x-3)$
$x(x-3)(x+1)$

$5x+3 = A(x-3)(x+1) + B(x)(x+1) + C(x)(x-3)$

$x=3: \quad 18 = B(3)(4)$
$B = \frac{18}{12} = \frac{3}{2}$

$x=-1: \quad -2 = C(-1)(-4)$
$-2 = 4C$
$C = -1/2$

$x=0: \quad 3 = A(-3)(1)$
$3 = -3A$
$A = -1$

$\int \left(\frac{-1}{x} + \frac{3}{2(x-3)} - \frac{1}{2(x+1)}\right) dx$

$= -\ln|x| + \frac{3}{2}\ln|x-3| - \frac{1}{2}\ln|x+1| + C$

$\ln\left|\frac{(x-3)^{3/2}}{(x+1)^{1/2}(x)}\right| + C$

5) *

$$\int \frac{x^4+8x^2+8}{x^3-4x}\,dx$$

$$\begin{array}{r|l} & x \\ \hline x^3-4x & x^4+0x^3+8x^2+0x+8 \\ & -\ (x^4 \qquad -4x^2) \\ \hline & 12x^2+8 \end{array}$$

$$\int x + \frac{12x^2+8}{x^3-4x}\,dx$$

$x(x^2-4)$

$$\frac{12x^2+8}{x(x-2)(x+2)} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{x+2}$$

$$12x^2+8 = A(x-2)(x+2) + B(x)(x+2) + C(x)(x-2)$$

$x=2$: $56 = B(2)(4)$ $\quad B=7$

$x=-2$: $56 = C(-2)(-4)$ $\quad C=7$

$x=0$: $8 = A(-2)(2)$ $\quad A=-2$

$$\int \left(x + \frac{-2}{x} + \frac{7}{x-2} + \frac{7}{x+2}\right) dx$$

$$\frac{1}{2}x^2 - 2\ln|x| + 7\ln|x-2| + 7\ln|x+2| + C$$

6) *

$$\int \frac{x-13}{2x^2-7x+3}\,dx \qquad \frac{x-13}{(2x-1)(x-3)} = \frac{A}{2x-1} + \frac{B}{x-3}$$

$$x-13 = A(x-3) + B(2x-1)$$

$x=3$: $-10 = B(5)$ $\quad B=-2$

$x=\frac{1}{2}$: $-12.5 = A(-2.5)$ $\quad A=5$

$$\int \left(\frac{5}{2x-1} + \frac{-2}{x-3}\right) dx$$

$u = 2x-1$

$du = 2dx$

$\frac{1}{2}du = dx$

$$= \frac{5}{2}\ln|2x-1| - 2\ln|x-3| + C$$

7) $\int \frac{x-12}{x^2-4x}\,dx \qquad \frac{x-12}{x(x-4)} = \frac{A}{x} + \frac{B}{x-4}$

$x-12 = A(x-4) + Bx$

$x=0: \quad -12 = -4A \qquad A = 3$

$x=4: \quad -8 = B(4) \qquad B = -2$

$\int \left(\frac{3}{x} + \frac{-2}{x-4}\right)dx = 3\ln|x| - 2\ln|x-4| + C$

$= \ln \frac{|x|^3}{(x-4)^2} + C$

8) $\int \frac{2x+16}{x^2+x-6}\,dx \qquad \frac{2x+16}{(x+3)(x-2)} = \frac{A}{x+3} + \frac{B}{x-2}$

$2x+16 = A(x-2) + B(x+3)$

$x=-3: \quad 10 = A(-5) \qquad A = -2$

$x=2: \quad 20 = B(5) \qquad B = 4$

$\int \left(\frac{-2}{x+3} + \frac{4}{x-2}\right)dx = -2\ln|x+3| + 4\ln|x-2| + C$

$\ln \frac{(x-2)^4}{(x+3)^2} + C$

9) $\int \frac{7}{2x^2-5x-3}\,dx \qquad \frac{7}{(2x+1)(x-3)} = \frac{A}{2x+1} + \frac{B}{x-3}$

$7 = A(x-3) + B(2x+1)$

$x=3: \quad 7 = 7B \qquad B = 1$

$x=-1/2: \quad 7 = -3,5A \qquad A = -2 \qquad -2\cdot\frac{1}{2}\ln|2x+1| + \ln|x-3| + C$

$\int \left(\frac{-2}{2x+1} + \frac{1}{x-3}\right)dx \qquad \ln\left|\frac{x-3}{2x+1}\right| + C$

$u = 2x+1$

$du = 2dx \qquad \frac{1}{2}du = dx$

10) $\int \frac{2}{x^2-4x+3}\,dx \quad \frac{2}{(x-1)(x-3)} = \frac{A}{x-1} + \frac{B}{x-3}$

$2 = A(x-3) + B(x-1)$

$x=1: \quad 2 = A(-2) \quad A = -1$

$x=3: \quad 2 = B(2) \quad B = 1$

$\int\left(\frac{-1}{x-1} + \frac{1}{x-3}\right)dx = -\ln|x-1| + \ln|x-3| + C$

$\ln\left|\frac{x-3}{x-1}\right| + C$

11) $\int \frac{x-1}{x(x-2)}\,dx \quad \frac{x-1}{x(x-2)} = \frac{A}{x} + \frac{B}{x-2}$

$x-1 = A(x-2) + Bx$

$x=0: \quad -1 = A(-2) \quad A = \frac{1}{2}$

$x=2: \quad 1 = B(2) \quad B = \frac{1}{2}$

$\int\left(\frac{1}{2}\cdot\frac{1}{x} + \frac{1}{2}\cdot\frac{1}{x-2}\right)dx = \frac{1}{2}\ln|x| + \frac{1}{2}\ln|x-2| + C$

$\frac{1}{2}\ln|x(x-2)| + C$

12) $\int \frac{2x^3-4x^2-x-3}{x^2-2x-3}\,dx$

$$\begin{array}{r} 2x \\ x^2-2x-3\,\overline{)\,2x^3-4x^2-x-3} \\ -\underline{(\cancel{2x^3}-\cancel{4x^2}-6x)} \\ 5x-3 \end{array}$$

$\int 2x + \frac{5x-3}{(x-3)(x+1)}\,dx$

$\longrightarrow \frac{5x-3}{(x-3)(x+1)} = \frac{A}{x-3} + \frac{B}{x+1}$

$5x-3 = A(x+1) + B(x-3)$

$x=3: \quad 12 = A(4) \quad A = 3$

$x=-1: \quad -8 = -4B \quad B = 2$

$\int\left(2x + \frac{3}{x-3} + \frac{2}{x+1}\right)dx \longrightarrow \frac{2x^3}{3} + 3\ln|x-3| + 2\ln|x+1| + C$

13) $\displaystyle\int \frac{2x^3}{x^2-4}\,dx$

$$\begin{array}{r|l} & \quad\quad\quad\quad\quad 2x \\ \hline x^2-4 & \cancel{2x}^3 + 0x^2 + 0x + 0 \\ - & \cancel{2x}^3 \quad\quad\quad -8x \\ \hline & \quad\quad\quad\quad\quad 8x \end{array}$$

$$\int\left(2x + \frac{8x}{x^2-4}\right)dx$$

$$\longrightarrow \frac{8x}{(x+2)(x-2)} = \frac{A}{x+2} + \frac{B}{x-2}$$

$$8x = A(x-2) + B(x+2)$$

$x=-2$: $-16 = A(-4)$ $\quad A = 4$

$x=2$: $16 = 4B$ $\quad B = 4$

$$\int\left(2x + \frac{4}{x+2} + \frac{4}{x-2}\right)dx$$

$$x^2 + 4\ln|x+2| + 4\ln|x-2| + C$$

$$x^2 + \ln(x+2)^4 + \ln(x-2)^4 + C$$

$$x^2 + \ln(x+2)^4(x-2)^4 + C$$

OR

$\displaystyle\int \frac{8x}{x^2-4}\,dx$ $\quad u = x^2-4$

$du = 2x\,dx$

$\frac{1}{2}du = x\,dx$

$$8\cdot\frac{1}{2}\int\frac{1}{u}\,du = 4\ln|u|$$

$$4\ln(x^2-4) + C$$

$$x^2 + \ln(x^2-4)^4 + C$$

14) $\displaystyle\int \frac{3x^4+1}{x^2-1}\,dx$

$$
\begin{array}{r|l}
 & 3x^2 \qquad + 3 \\
\hline
x^2-1 & 3x^4 + 0x^3 + 0x^2 + 0x + 1 \\
 & -(3x^4 \qquad - 3x^2) \\
\hline
 & 3x^2 + 1 \\
 & -(3x^2 - 3) \\
\hline
 & 4
\end{array}
$$

$$\int \left(3x^2+3+\frac{4}{x^2-1}\right)dx$$

$$\frac{4}{(x+1)(x-1)} = \frac{A}{x+1} + \frac{B}{x-1}$$

$4 = A(x-1) + B(x+1)$

$x=1: \; 4 = 2B \qquad B = 2$

$x=-1: \; 4 = A(-2) \quad A = -2$

$$\int 3x^2+3+\frac{-2}{x+1}+\frac{2}{x-1}$$

$$x^3 + 3x + 2\ln|x-1| - 2\ln|x+1| + C$$

$$x^3 + 3x + 2\ln\left|\frac{x-1}{x+1}\right| + C$$