MC/FRQ Wkst KEY

5. 2002 #5

(a) $\frac{dy}{dx} = 0$ when $x = 3$

$\left.\frac{d^2y}{dx^2}\right|_{(3,-2)} = \left.\frac{-y - y'(3-x)}{y^2}\right|_{(3,-2)} = \frac{1}{2}$,

so f has a local minimum at this point.

or

Because f is continuous for $1 < x < 5$, there is an interval containing $x = 3$ on which $y < 0$. On this interval, $\frac{dy}{dx}$ is negative to the left of $x = 3$ and $\frac{dy}{dx}$ is positive to the right of $x = 3$. Therefore f has a local minimum at $x = 3$.

3 {
- 1 : $x = 3$
- 1 : local minimum
- 1 : justification

(b) $y\,dy = (3 - x)\,dx$

$\frac{1}{2}y^2 = 3x - \frac{1}{2}x^2 + C$

$8 = 18 - 18 + C$; $C = 8$

$y^2 = 6x - x^2 + 16$

$y = -\sqrt{6x - x^2 + 16}$

6 {
- 1 : separates variables
- 1 : antiderivative of dy term
- 1 : antiderivative of dx term
- 1 : constant of integration
- 1 : uses initial condition $g(6) = -4$
- 1 : solves for y

Note: max 3/6 [1-1-1-0-0-0] if no constant of integration

Note: 0/6 if no separation of variables

I. Euler's Method – Multiple Choice

5. C

$y \approx y(a) + \frac{dy}{dx}(dx)$

$y(1) = 2$
$y(1.5) = y(1) + (3)(.5) = 2 + 1.5 = 3.5$ $\boxed{C}$
$y(2) = y(1.5) + (5)(.5) = 3.5 + 2.5 = 6$

7. D Given that $y(1) = -3$ and $\frac{dy}{dx} = 2x + y$, what is the approximation for $y(2)$ if Euler's method is used with a step size of 0.5, starting at $x = 1$?

$y(1.5) = y(1) + (0.5)(2(1) + (-3)) = -3 + (0.5)(-1) = -3.5$
$y(2) = y(1.5) + (0.5)(2(1.5) + (-3.5)) = -3.5 + (0.5)(-0.5) = -3.75$ $\boxed{D}$

II. Euler's Method – Free Response

1. 2001 BC5 Part b

Let f be the function satisfying $f'(x) = -3xf(x)$, for all real numbers x, with $f(1) = 4$

(b) $f(1.5) \approx f(1) + f'(1)(0.5)$ $= 4 - 3(1)(4)(0.5) = -2$ $f(2) \approx -2 + f'(1.5)(0.5)$ $\approx -2 - 3(1.5)(-2)(0.5) = 2.5$	2 : { 1 : Euler's method equations or equivalent table 1 : Euler approximation to $f(2)$ (not eligible without first point)

2. 2007 BC5 Form B Parts c and d

(c) $f\left(\frac{1}{2}\right) \approx f(0) + f'(0) \cdot \frac{1}{2} = -2 + (-3) \cdot \frac{1}{2} = -\frac{7}{2}$ $f'\left(\frac{1}{2}\right) \approx 3\left(\frac{1}{2}\right) + 2\left(-\frac{7}{2}\right) + 1 = -\frac{9}{2}$ $f(1) \approx f\left(\frac{1}{2}\right) + f'\left(\frac{1}{2}\right) \cdot \frac{1}{2} = -\frac{7}{2} + \left(-\frac{9}{2}\right) \cdot \frac{1}{2} = -\frac{23}{4}$	2 : { 1 : Euler's method with 2 steps 1 : Euler's approximation for $f(1)$
(d) $g'(0) = 3 \cdot 0 + 2 \cdot k + 1 = 2k + 1$ $g(1) \approx g(0) + g'(0) \cdot 1 = k + (2k + 1) = 3k + 1 = 0$ $k = -\frac{1}{3}$	2 : { 1 : $g(0) + g'(0) \cdot 1$ 1 : value of k

III. Logistic Growth Functions – Multiple Choice

21. B $\lim_{t\to\infty} M(t)$ is the maximum value of M, the carrying capacity which, in this problem, is 200 since the differential equation is of the logistic form $\frac{dM}{dt} = kM\left(1-\frac{M}{A}\right)$ where A is the carrying capacity. $\boxed{B}$

24. A

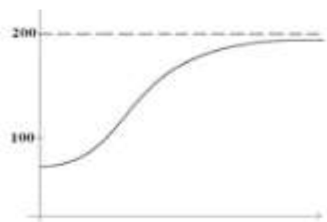
Using the following model formula for logistic growth: $\frac{dP}{dt} = kP(A-P)$

$\frac{dP}{dt} = kP(A-P) = kP(200-P) = k(200P-P^2)$

Only one of the 5 choices can be put into the form $k(200P-P^2)$:

$\frac{dP}{dt} = .2P - .001P^2 = .001(200P-P^2) \rightarrow k = .001$

$\boxed{A}$



Carrying Capacity = A = 200

Exponential Growth and Decay

1988 BC43:

43. A This is an example of exponential growth, $B = B_0 \cdot 2^{t/3}$. Find the value of t so $B = 3B_0$.

$$3B_0 = B_0 \cdot 2^{t/3} \Rightarrow 3 = 2^{t/3} \Rightarrow \ln 3 = \frac{t}{3}\ln 2 \Rightarrow t = \frac{3\ln 3}{\ln 2}$$

1993 AB42:

B This is an example of exponential growth. We know from pre-calculus that $w = 2\left(\frac{3.5}{2}\right)^{\frac{t}{2}}$ is an exponential function that meets the two given conditions. When $t = 3$, $w = 4.630$. Using calculus the student may translate the statement “increasing at a rate proportional to its weight” to mean exponential growth and write the equation $w = 2e^{kt}$. Using the given conditions, $3.5 = 2e^{2k}$; $\ln(1.75) = 2k$; $k = \frac{\ln(1.75)}{2}$; $w = 2e^{t\cdot\frac{\ln(1.75)}{2}}$. When $t = 3$, $w = 4.630$.

1993 BC 38:

C $\frac{dN}{dt} = kN \Rightarrow N = Ce^{kt}$. $N(0) = 1000 \Rightarrow C = 1000$. $N(7) = 1200 \Rightarrow k = \frac{1}{7}\ln(1.2)$. Therefore $N(12) = 1000e^{\frac{12}{7}\ln(1.2)} \approx 1367$.

1998 AB 84:

A A known solution to this differential equation is $y(t) = y(0)e^{kt}$. Use the fact that the population is $2y(0)$ when $t = 10$. Then
$2y(0) = y(0)e^{k(10)} \Rightarrow e^{10k} = 2 \Rightarrow k = (0.1)\ln 2 = 0.069$

FRQ Wkst KEY

2007 BC5 Form B:

(a) $\frac{d^2y}{dx^2} = 3 + 2\frac{dy}{dx} = 3 + 2(3x + 2y + 1) = 6x + 4y + 5$

$2: \begin{cases} 1: 3 + 2\frac{dy}{dx} \\ 1: \text{answer} \end{cases}$

(b) If $y = mx + b + e^{rx}$ is a solution, then
$m + re^{rx} = 3x + 2\left(mx + b + e^{rx}\right) + 1.$

If $r \neq 0$: $m = 2b + 1,\ r = 2,\ 0 = 3 + 2m,$
so $m = -\frac{3}{2},\ r = 2,$ and $b = -\frac{5}{4}.$

OR

If $r = 0$: $m = 2b + 3,\ r = 0,\ 0 = 3 + 2m,$
so $m = -\frac{3}{2},\ r = 0,\ b = -\frac{9}{4}.$

$3: \begin{cases} 1: \frac{dy}{dx} = m + re^{rx} \\ 1: \text{value for } r \\ 1: \text{values for } m \text{ and } b \end{cases}$

(c) $f\left(\frac{1}{2}\right) \approx f(0) + f'(0) \cdot \frac{1}{2} = -2 + (-3) \cdot \frac{1}{2} = -\frac{7}{2}$
$f'\left(\frac{1}{2}\right) \approx 3\left(\frac{1}{2}\right) + 2\left(-\frac{7}{2}\right) + 1 = -\frac{9}{2}$
$f(1) \approx f\left(\frac{1}{2}\right) + f'\left(\frac{1}{2}\right) \cdot \frac{1}{2} = -\frac{7}{2} + \left(-\frac{9}{2}\right) \cdot \frac{1}{2} = -\frac{23}{4}$

$2: \begin{cases} 1: \text{Euler's method with 2 steps} \\ 1: \text{Euler's approximation for } f(1) \end{cases}$

(d) $g'(0) = 3 \cdot 0 + 2 \cdot k + 1 = 2k + 1$
$g(1) \approx g(0) + g'(0) \cdot 1 = k + (2k + 1) = 3k + 1 = 0$
$k = -\frac{1}{3}$

$2: \begin{cases} 1: g(0) + g'(0) \cdot 1 \\ 1: \text{value of } k \end{cases}$

2013 BC5:

(a) $\lim_{x\to 0}(f(x)+1) = -1+1 = 0$ and $\lim_{x\to 0}\sin x = 0$

Using L'Hospital's Rule,

$$\lim_{x\to 0}\frac{f(x)+1}{\sin x} = \lim_{x\to 0}\frac{f'(x)}{\cos x} = \frac{f'(0)}{\cos 0} = \frac{(-1)^2\cdot 2}{1} = 2$$

$2 : \begin{cases} 1 : \text{L'Hospital's Rule} \\ 1 : \text{answer} \end{cases}$

(b) $$f\left(\frac{1}{4}\right) \approx f(0) + f'(0)\left(\frac{1}{4}\right)$$
$$= -1 + (2)\left(\frac{1}{4}\right) = -\frac{1}{2}$$

$$f\left(\frac{1}{2}\right) \approx f\left(\frac{1}{4}\right) + f'\left(\frac{1}{4}\right)\left(\frac{1}{4}\right)$$
$$= -\frac{1}{2} + \left(-\frac{1}{2}\right)^2\left(2\cdot\frac{1}{4}+2\right)\left(\frac{1}{4}\right) = -\frac{11}{32}$$

$2 : \begin{cases} 1 : \text{Euler's method} \\ 1 : \text{answer} \end{cases}$

(c) $$\frac{dy}{dx} = y^2(2x+2)$$
$$\frac{dy}{y^2} = (2x+2)\,dx$$
$$\int\frac{dy}{y^2} = \int(2x+2)\,dx$$
$$-\frac{1}{y} = x^2+2x+C$$
$$-\frac{1}{-1} = 0^2+2\cdot 0+C \Rightarrow C = 1$$
$$-\frac{1}{y} = x^2+2x+1$$
$$y = -\frac{1}{x^2+2x+1} = -\frac{1}{(x+1)^2}$$

$5 : \begin{cases} 1 : \text{separation of variables} \\ 1 : \text{antiderivatives} \\ 1 : \text{constant of integration} \\ 1 : \text{uses initial condition} \\ 1 : \text{solves for } y \end{cases}$

Note: max $2/5$ [1-1-0-0-0] if no constant of integration

Note: $0/5$ if no separation of variables

2004 BC5:

(a) For this logistic differential equation, the carrying capacity is 12.

If $P(0) = 3$, $\lim_{t\to\infty} P(t) = 12$.

If $P(0) = 20$, $\lim_{t\to\infty} P(t) = 12$.

$2 : \begin{cases} 1 : \text{answer} \\ 1 : \text{answer} \end{cases}$

(b) The population is growing the fastest when P is half the carrying capacity. Therefore, P is growing the fastest when $P = 6$.

1 : answer

(c) $\frac{1}{Y}dY = \frac{1}{5}\left(1 - \frac{t}{12}\right)dt = \left(\frac{1}{5} - \frac{t}{60}\right)dt$

$\ln|Y| = \frac{t}{5} - \frac{t^2}{120} + C$

$Y(t) = Ke^{\frac{t}{5} - \frac{t^2}{120}}$

$K = 3$

$Y(t) = 3e^{\frac{t}{5} - \frac{t^2}{120}}$

5 :
- 1 : separates variables
- 1 : antiderivatives
- 1 : constant of integration
- 1 : uses initial condition
- 1 : solves for Y
  - 0/1 if Y is not exponential

Note: max 2/5 [1-1-0-0-0] if no constant of integration

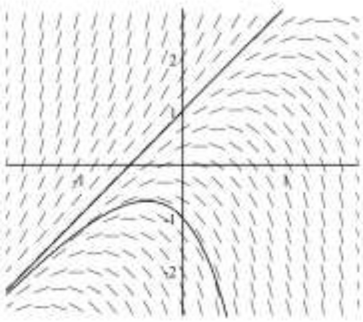
Note: 0/5 if no separation of variables

(d) $\lim_{t\to\infty} Y(t) = 0$

1 : answer
0/1 if Y is not exponential

2002 BC5:

(a)



(b) $f(0.1) \approx f(0) + f'(0)(0.1)$

$= 1 + (2 - 0)(0.1) = 1.2$

$f(0.2) \approx f(0.1) + f'(0.1)(0.1)$

$\approx 1.2 + (2.4 - 0.4)(0.1) = 1.4$

(c) Substitute $y = 2x + b$ in the DE:

$2 = 2(2x + b) - 4x = 2b$, so $b = 1$

OR

Guess $b = 1$, $y = 2x + 1$

Verify: $2y - 4x = (4x + 2) - 4x = 2 = \frac{dy}{dx}$.

(d) g has local maximum at $(0, 0)$.

$g'(0) = \left.\frac{dy}{dx}\right|_{(0,0)} = 2(0) - 4(0) = 0$, and

$g''(x) = \frac{d^2y}{dx^2} = 2\frac{dy}{dx} - 4$, so

$g''(0) = 2g'(0) - 4 = -4 < 0$.

(a) 2 { 1: solution curve through $(0,1)$; 1: solution curve through $(0,-1)$ }

Curves must go through the indicated points, follow the given slope lines, and extend to the boundary of the slope field.

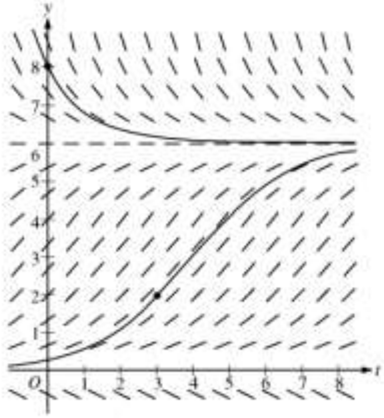
(b) 2 { 1: Euler's method equations or equivalent table applied to (at least) two iterations; 1: Euler approximation to $f(0.2)$ (not eligible without first point) }

(c) 2 { 1: uses $\frac{d}{dx}(2x + b) = 2$ in DE; 1: $b = 1$ }

(d) 3 { 1: $g'(0) = 0$; 1: shows $g''(0) = -4$; 1: conclusion }

2008 BC6: omit part c

(a)



2 : { 1: solution curve through $(0,8)$
1: solution curve through $(3,2)$ }

(b) $f\left(\frac{1}{2}\right) = 8 + (-2)\left(\frac{1}{2}\right) = 7$

$f(1) = 7 + \left(-\frac{7}{8}\right)\left(\frac{1}{2}\right) = \frac{105}{16}$

2 : { 1 : Euler's method with two steps
1 : approximation of $f(1)$ }

(c) $\frac{d^2y}{dt^2} = \frac{1}{8}\frac{dy}{dt}(6 - y) + \frac{y}{8}\left(-\frac{dy}{dt}\right)$

$f(0) = 8$; $f'(0) = \left.\frac{dy}{dt}\right|_{t=0} = \frac{8}{8}(6 - 8) = -2$; and

$f''(0) = \left.\frac{d^2y}{dt^2}\right|_{t=0} = \frac{1}{8}(-2)(-2) + \frac{8}{8}(2) = \frac{5}{2}$

The second-degree Taylor polynomial for f about $t = 0$ is $P_2(t) = 8 - 2t + \frac{5}{4}t^2$.

$f(1) \approx P_2(1) = \frac{29}{4}$

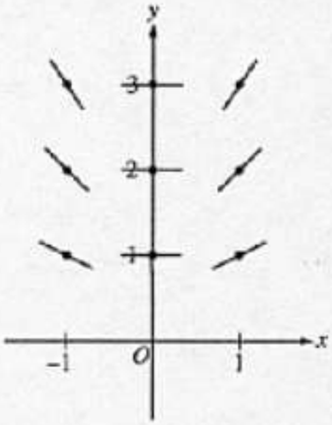
4 : { 2 : $\frac{d^2y}{dt^2}$
1 : second-degree Taylor polynomial
1 : approximation of $f(1)$ }

(d) The range of f for $t \ge 0$ is $6 < y \le 8$.

1 : answer

1998 BC4:

(a)



1: line segments at nine points with negative - zero - positive slope left to right and increasing steepness bottom to top at $x = 1$ and $x = -1$

(b) $f(0.1) \approx f(0) + f'(0)(0.1)$

$= 3 + \frac{1}{2}(0)(3)(0.1) = 3$

$f(0.2) \approx f(0.1) + f'(0.1)(0.1)$

$= 3 + \frac{1}{2}(0.1)(3)(0.1)$

$= 3 + \frac{.03}{2} = 3.015$

2 { 1: Euler's Method equations or equivalent table; 1: answer (not eligible without first point) }

Special Case: 1/2 for first iteration 3.015 and second iteration 3.045

(c) $\frac{dy}{dx} = \frac{xy}{2}$

$\int \frac{dy}{y} = \int \frac{x}{2}\,dx$

$\ln|y| = \frac{1}{4}x^2 + C_1$

$y = Ce^{x^2/4}$

$3 = Ce^0 \implies C = 3$

$y = 3e^{x^2/4}$

$f(0.2) = 3e^{.04/4} = 3e^{.01} = 3.030$

6 {
1: separates variables
1: antiderivative of dy term
1: antiderivative of dx term
1: solves for y
1: solves for constant of integration
1: evaluates $f(0.2)$
}

Note: max 4/6 [1-1-1-0-0-1] if no constant of integration