10.5 Area in Polar Coordinates

* If f is continuous and non-negative on the interval $[\alpha, \beta]$, $0 < \beta - \alpha \leq 2\pi$, then the area of the region bounded by the graph of $r = f(\theta)$ between radial lines $\theta = \alpha$ and $\theta = \beta$ is:

$$A = \frac{1}{2}\int_{\alpha}^{\beta} \underbrace{[f(\theta)]^2}_{r^2}\, d\theta$$

$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$
$\cos^2 x = \frac{1}{2}(1 + \cos 2x)$

* Determining Radial Lines:

1) Sketch region whose area is to be determined
2) Draw an arbitrary "radial line" from the pole to boundary curve
3) Ask, "Over what interval of values must θ vary in order for the radial line to sweep out the region R?"
4) Ans from #3 determines limits of integration

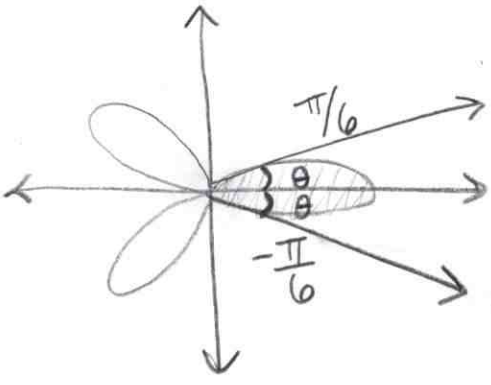
* Tangent lines at the Pole: If $f(\alpha) = 0$ and $f'(\alpha) \neq 0$, then $\theta = \alpha$ is tangent at the pole to graph of $r = f(\theta)$

Sketch graph, shade region, find area. (Non-Calc)

(1) One petal of $r = 2\cos(3\theta)$

$n = 3$

1



$2\cos(3\theta) = 0$

$\cos(3\theta) = 0$

$3\theta = \frac{\pi}{2}$ $\quad 3\theta = \frac{3\pi}{2}$

$\boxed{\theta = \frac{\pi}{6}}$ $\quad \theta = \frac{\pi}{2}$

$$A = \frac{1}{2}\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} (2\cos 3\theta)^2 \, d\theta$$

symmetry:

$$A = \int_{0}^{\frac{\pi}{6}} (2\cos 3\theta)^2 \, d\theta$$

$\frac{1}{2}(1+\cos 2\theta)$

$$A = 4\int_{0}^{\frac{\pi}{6}} \cos^2(3\theta) \, d\theta$$

$\cos^2\theta = \frac{1+\cos 2\theta}{2}$

$$A = 4\int_{0}^{\frac{\pi}{6}} \frac{1+\cos 6\theta}{2} \, d\theta$$

$$A = 2\int_{0}^{\frac{\pi}{6}} (1+\cos 6\theta) \, d\theta$$

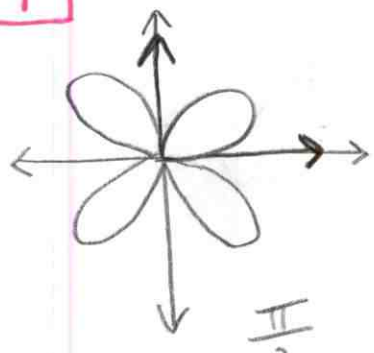
$$2\left[\theta + \frac{1}{6}\sin 6\theta \,\Big|_{0}^{\frac{\pi}{6}}\right]$$

$\rightarrow 2\left[\frac{\pi}{6} + \frac{1}{6}\sin\pi - \left(0 + \frac{1}{6}\sin 0\right)\right]$

$\frac{2\pi}{6} = \boxed{\frac{\pi}{3}}$

(2) One petal of $r = 4\sin(2\theta)$

2YT



$4\sin 2\theta = 0$

$\sin 2\theta = 0$

$2\theta = 0 \qquad 2\theta = \pi$

$\theta = 0 \qquad \theta = \frac{\pi}{2}$

$$A = \frac{1}{2}\int_0^{\frac{\pi}{2}} [4\sin 2\theta]^2 \, d\theta$$

$$= 8\int_0^{\frac{\pi}{2}} \sin^2 2\theta \, d\theta$$

$$\sin^2\theta = \frac{1-\cos 2\theta}{2}$$

$$= \frac{1}{2}(1-\cos 4\theta)$$

$$= 8\int_0^{\frac{\pi}{2}} \frac{1-\cos 4\theta}{2} \, d\theta$$

$$= 4\int_0^{\frac{\pi}{2}} (1-\cos 4\theta) \, d\theta$$

$$4\left[\theta - \frac{1}{4}\sin 4\theta \Big|_0^{\frac{\pi}{2}}\right]$$

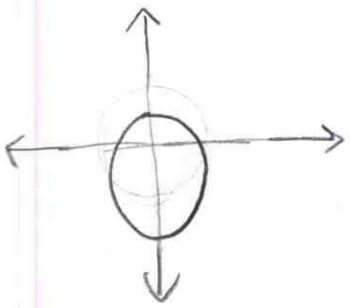
$$4\left(\frac{\pi}{2} - \frac{1}{4}\cancel{\sin 2\pi}^{0} - (0-0)\right)$$

$$4\left(\frac{\pi}{2}\right) = \boxed{2\pi}$$

(4) Interior of $r = 2 - \sin\theta$

3

$[0, 2\pi]$



$$2 - \sin\theta = 0$$
$$-\sin\theta = -2$$
$$\sin\theta = 2 \qquad \theta = \phi$$

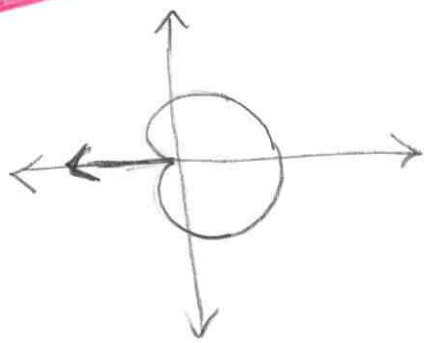
↳ go to period

$$A = \frac{1}{2}\int_0^{2\pi} (2 - \sin\theta)^2 \, d\theta$$

$$A = \int_0^{\pi} (2 - \sin\theta)^2 \, d\theta$$

(3) Interior of $r = 2 + 2\cos\theta$

4-YT



$$2 + 2\cos\theta = 0$$
$$2\cos\theta = -2$$
$$\cos\theta = -1$$
$$\theta = \pi$$

$$A = \frac{1}{2}\int_{-\pi}^{\pi} (2+2\cos\theta)^2 \, d\theta$$

$$A = \int_{0}^{\pi} (2+2\cos\theta)^2 \, d\theta$$

$(2+2\cos\theta)(2+2\cos\theta)$

$4 + 8\cos\theta + 4\cos^2\theta$

$$= \int_{0}^{\pi} (4 + 8\cos\theta + 4\cos^2\theta)\, d\theta$$

$$= 4\int_{0}^{\pi} (1 + 2\cos\theta + \cos^2\theta)\, d\theta$$

$\cos^2\theta \rightarrow \frac{1}{2} - \frac{1}{2}\cos 2\theta$

$$= 4\int_{0}^{\pi} \left(1 + 2\cos\theta + \frac{1-\cos 2\theta}{2}\right) d\theta$$

$$4\left[\theta + 2\sin\theta + \frac{1}{2}\theta + \frac{1}{4}\sin 2\theta\right]\Big|_0^{\pi}$$

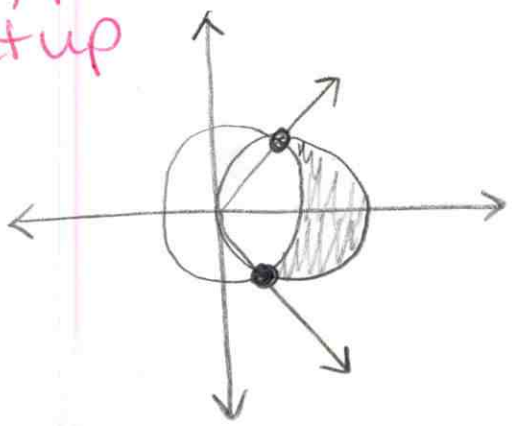
$$4\left[\pi + 0 + \frac{\pi}{2} + 0 - (0)\right]$$

$$4\left(\frac{3\pi}{2}\right) = \boxed{6\pi}$$

Calc Active:

(5) Inside $r = 3\cos\theta$ (circle), Outside $r = 2 - \cos\theta$ (limacon)

7 YT setup

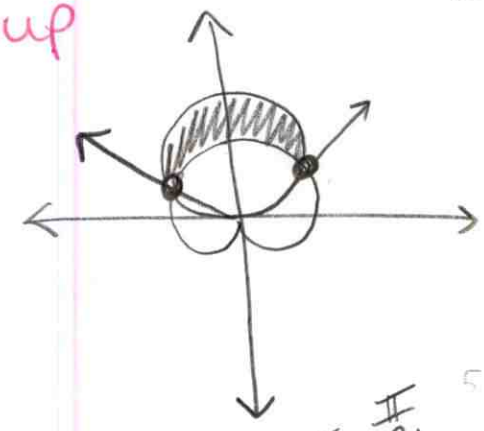


$$3\cos\theta = 2 - \cos\theta$$
$$4\cos\theta = 2$$
$$\cos\theta = \frac{1}{2}$$
$$\theta = \frac{\pi}{3}, \frac{5\pi}{3}$$

$$A = 2\left[\frac{1}{2}\int_0^{\pi/3}(3\cos\theta)^2 d\theta - \frac{1}{2}\int_0^{\pi/3}(2-\cos\theta)^2 d\theta\right]$$

(6) Inside $r = 3\sin\theta$ (circle), Outside $r = 1 + \sin\theta$ (cardiod)

8 YT setup



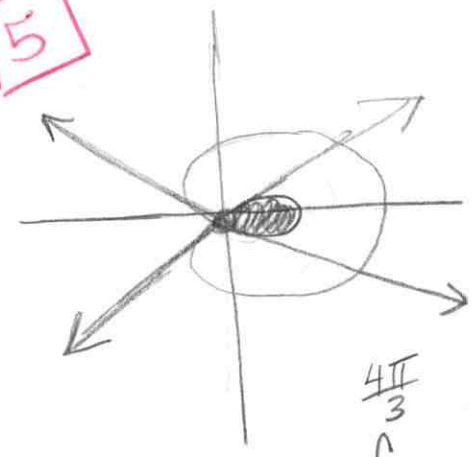
$$3\sin\theta = 1 + \sin\theta$$
$$2\sin\theta = 1$$
$$\sin\theta = \frac{1}{2}$$
$$\theta = \frac{\pi}{6}, \frac{5\pi}{6}$$

$$2\left[\frac{1}{2}\int_{\pi/6}^{\pi/2}(3\sin\theta)^2 d\theta - \frac{1}{2}\int_{\pi/6}^{\pi/2}(1+\sin\theta)^2 d\theta\right]$$

(7) Area of inner loop: $r = 1 + 2\cos\theta$

5

limacon



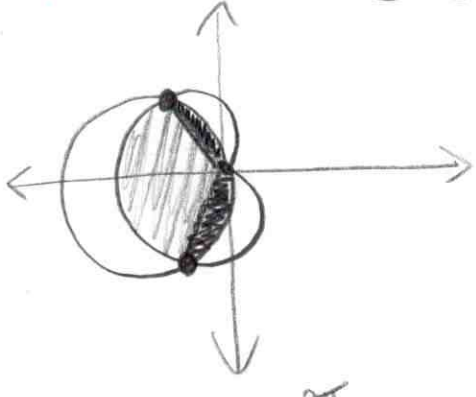
$$1 + 2\cos\theta = 0$$
$$2\cos\theta = -1$$
$$\cos\theta = -\frac{1}{2}$$
$$\theta = \frac{2\pi}{3}, \frac{4\pi}{3}$$

$$A = \frac{1}{2}\int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} (1 + 2\cos\theta)^2 \, d\theta \approx .544$$

(8) Area of region common

9YT to setup

$r = -6\cos\theta$ circle $\qquad r = 2 - 2\cos\theta$



$$-6\cos\theta = 2 - 2\cos\theta$$
$$-4\cos\theta = 2$$
$$\cos\theta = -\frac{1}{2}$$
$$\theta = \frac{2\pi}{3}, \frac{4\pi}{3}$$

bananas:

$$A = 2 \cdot \frac{1}{2}\int_{\frac{\pi}{2}}^{\frac{2\pi}{3}} (-6\cos\theta)^2 \, d\theta = 1.631$$

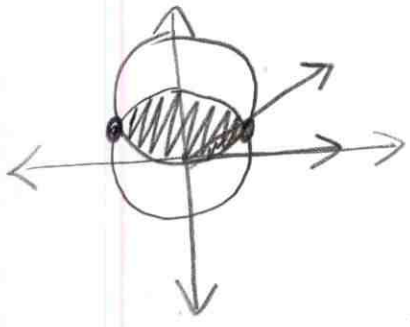
$+$ $\qquad\qquad +$ $\quad = 15.708$

pizza:

$$A = \frac{1}{2}\int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} (2 - 2\cos\theta)^2 \, d\theta = 14.077$$

(9) (6) Common interior of $r = 4\sin\theta$ and $r = 2$

$4\sin\theta = 0$
$\sin\theta = 0$
$\theta = 0$



$4\sin\theta = 2$
$\sin\theta = \frac{1}{2}$
$\theta = \frac{\pi}{6}, \frac{5\pi}{6}$

$$A = 2\left[\frac{1}{2}\int_0^{\frac{\pi}{6}} (4\sin\theta)^2 d\theta + \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 2^2 d\theta\right]$$

$$A = 16\int_0^{\frac{\pi}{6}} \sin^2\theta\, d\theta + 4\theta\Big|_{\frac{\pi}{6}}^{\frac{\pi}{2}}$$

$$A = 16\cdot\frac{1}{2}\int_0^{\frac{\pi}{6}} (1-\cos 2\theta)\, d\theta + 4\theta\Big|_{\pi/6}^{\pi/2}$$

$$A = 8\left(\theta - \frac{1}{2}\sin 2\theta\right)\Big|_0^{\frac{\pi}{6}} + 4\theta\Big|_{\frac{\pi}{6}}^{\frac{\pi}{2}}$$

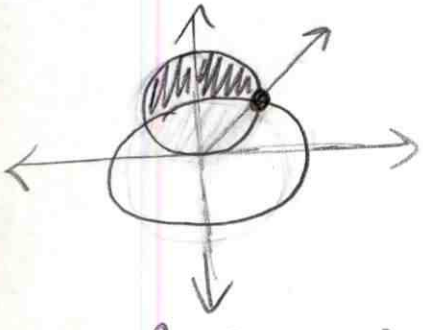
$$A = 8\left[\left(\frac{\pi}{6} - \frac{\sqrt{3}}{4}\right) - (0)\right] + \left(2\pi - \frac{2\pi}{3}\right)$$

$$\frac{4\pi}{3} - \frac{8\sqrt{3}}{4} + 2\pi - \frac{2\pi}{3}$$

$$\boxed{\frac{8\pi}{3} - 2\sqrt{3}}$$

10 YT setup

10) inside $r = 3\sin\theta$
outside $r = 2 - \sin\theta$



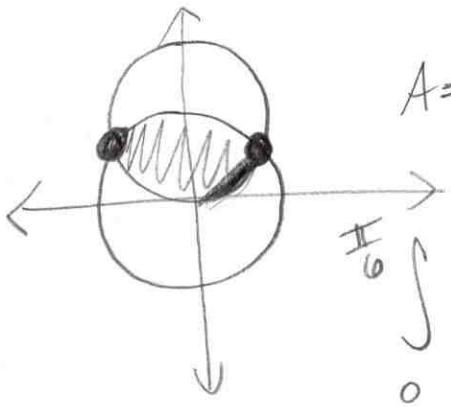
$3\sin\theta = 2 - \sin\theta$
$4\sin\theta = 2$
$\sin\theta = \frac{1}{2}$
$\theta = \frac{\pi}{6}$

$$2\left[\frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (3\sin\theta)^2 d\theta - \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (2-\sin\theta)^2 d\theta\right]$$

$$\vdots$$

$$\boxed{3\sqrt{3}}$$

2 common interior $r=4\sin\theta$ and $r=2$



$$A=2\left[\frac{1}{2}\int_0^{\frac{\pi}{6}}(4\sin\theta)^2d\theta+\frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}2^2d\theta\right]$$

$$\int_0^{\frac{\pi}{6}}16\sin^2\theta\,d\theta+\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}4d\theta$$

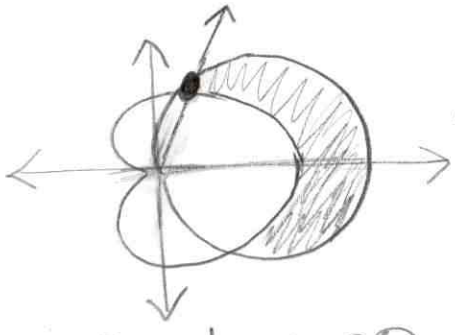
$4\sin\theta=2$
$\sin\theta=\frac{1}{2}$
$\theta=\frac{\pi}{6}$

$$16\cdot\frac{1}{2}\int_0^{\frac{\pi}{6}}(1-\cos2\theta)d\theta$$

$$8\left[\theta-\frac{1}{2}\sin2\theta\right]\Big|_0^{\frac{\pi}{6}}+4\theta\Big|_{\frac{\pi}{6}}^{\frac{\pi}{2}}$$

$$8\left[\left(\frac{\pi}{6}-\frac{\sqrt{3}}{4}\right)-0\right]+2\pi-\frac{2\pi}{3}=\frac{8\pi}{3}-2\sqrt{3}$$

4 inside $r=3\cos\theta$, outside $r=1+\cos\theta$



$$A=2\left[\frac{1}{2}\int_0^{\frac{\pi}{6}}(3\cos\theta)^2d\theta-\frac{1}{2}\int_0^{\frac{\pi}{6}}(1+\cos\theta)^2d\theta\right]$$

$3\cos\theta=1+\cos\theta$
$2\cos\theta=1$
$\cos\theta=\frac{1}{2}$
$\theta=\frac{\pi}{3}$