AP Calculus BC Name: Key

10.5 Area in Polar Coordinates

Warm-Up: Use $\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{f(\theta)\cos\theta + f'(\theta)\sin\theta}{-f(\theta)\sin\theta + f'(\theta)\cos\theta}$

$r' = -\cos\theta$

1. Find θ of the horizontal tangents to $r = 1 - \sin\theta$.

$(1-\sin\theta)\cos\theta - \cos\theta\sin\theta = 0$

$\cos\theta - \cos\theta\sin\theta - \cos\theta\sin\theta = 0$

$\cos\theta - 2\cos\theta\sin\theta = 0$

$\cos\theta(1 - 2\sin\theta) = 0$

$\cos\theta = 0$ $\quad$ $\sin\theta = \frac{1}{2}$

$\theta = \cancel{\frac{\pi}{2}}, \frac{3\pi}{2}$ $\quad$ $\theta = \frac{\pi}{6}, \frac{5\pi}{6}$

2. Find θ of the vertical tangents to $r = 1 - \sin\theta$.

$-(1-\sin\theta)\sin\theta - \cos\theta\cos\theta = 0$

$-\sin\theta + \sin^2\theta - \cos^2\theta = 0$

$-\sin\theta + \sin^2\theta - (1 - \sin^2\theta) = 0$

$-\sin\theta + \sin^2\theta - 1 + \sin^2\theta = 0$

$2\sin^2\theta - \sin\theta - 1 = 0$

$(2\sin\theta + 1)(\sin\theta - 1) = 0$

$\sin\theta = -\frac{1}{2}$ $\quad$ $\sin\theta = 1$

$\theta = \frac{7\pi}{6}, \frac{11\pi}{6}$ $\quad$ $\theta = \cancel{\pi/2}$

$r' = -3\sin\theta$

3. Find the points of horizontal tangency for $r = 2\csc\theta + 3$. $\quad$ $r' = -2\csc\theta\cot\theta$

$(2\csc\theta + 3)\cos\theta + (-2\csc\theta\cot\theta)\sin\theta = 0$

$\cancel{2\frac{\cos\theta}{\sin\theta}} + 3\cos\theta - \cancel{2\cot\theta} = 0$

$3\cos\theta = 0$

$\theta = \frac{\pi}{2}, \frac{3\pi}{2}$

$\theta = \frac{\pi}{2} : r = 5$ $\quad$ $\theta = \frac{3\pi}{2} : r = 1$

4. Find the points of vertical tangency for $r = 3\cos\theta$

$-3\cos\theta\sin\theta - 3\sin\theta\cos\theta = 0$

$-6\cos\theta\sin\theta = 0$

$\cos\theta = 0$ $\quad$ $\sin\theta = 0$

$\theta = \frac{\pi}{2}, \frac{3\pi}{2}$ $\quad$ $\theta = 0, \pi$

$\theta = \frac{\pi}{2} : r = 0$ $\quad$ $\theta = 0 : r = 3$

$\theta = \frac{3\pi}{2} : r = 0$ $\quad$ $\theta = \pi : r = -3$

Area in Polar Coordinates:

If f is continuous and non-negative on the interval $[\alpha, \beta]$, where $0 < \beta - \alpha < 2\pi$, then the area of the region bounded by the graph of $r = f(\theta)$ between radial lines $\theta = \alpha$ and $\theta = \beta$ is:

$$\text{Area} = \frac{1}{2}\int_{\alpha}^{\beta} [f(\theta)]^2 \, d\theta$$

Determining Radial Lines:

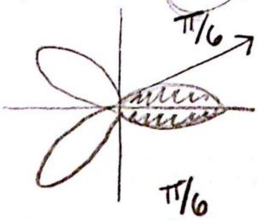
1) Sketch region whose area is to be determined.
2) Draw an arbitrary "radial line" from the pole to the boundary curve.
3) Ask, "Over what interval of values must θ vary in order for the radial line to sweep out the region R?"
4) Your answer for #3 determines the limits of integration.

Need to Know:

$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$ $\quad$ $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$

Sketch a graph, shade the region described, and find the area.

1. (Calc Inactive) One petal of $r = 2\cos(3\theta)$



$2\cos(3\theta) = 0$

$\cos(3\theta) = 0$

$3\theta = \frac{\pi}{2} \quad 3\theta = \frac{3\pi}{2}$

$\theta = \frac{\pi}{6} \quad \theta = \frac{\pi}{2}$

$A = 2 \cdot \frac{1}{2} \int_0^{\pi/6} (2\cos 3\theta)^2 \, d\theta$

$A = \int_0^{\pi/6} 4(\cos 3\theta)^2 \, d\theta$

$A = 4 \int_0^{\pi/6} \frac{1}{2}(1 + \cos 6\theta) \, d\theta$

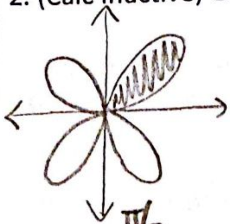
$A = 2 \int_0^{\pi/6} (1 + \cos 6\theta) \, d\theta$

$= 2\left(\theta + \frac{1}{6}\sin 6\theta\right)\Big|_0^{\pi/6}$

$= 2\left[\left(\frac{\pi}{6} + \cancel{\frac{1}{6}\sin\pi}\right) - \cancel{\left(0 + \frac{1}{6}\sin 0\right)}\right]$ (0, 0)

$= \boxed{\pi/3}$

2. (Calc Inactive) One petal of $r = 4\sin(2\theta)$



$4\sin(2\theta) = 0$

$\sin(2\theta) = 0$

$2\theta = 0 \quad 2\theta = \pi$

$\theta = 0 \quad \theta = \frac{\pi}{2}$

$A = \frac{1}{2} \int_0^{\pi/2} [4\sin 2\theta]^2 \, d\theta$

$8 \int_0^{\pi/2} (\sin^2 2\theta) \, d\theta$

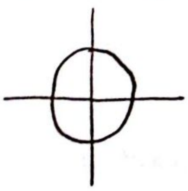
$8 \int_0^{\pi/2} \frac{1}{2}(1 - \cos 4\theta) \, d\theta$

$4 \int_0^{\pi/2} (1 - \cos 4\theta) \, d\theta$

$4\left(\theta - \frac{1}{4}\sin 4\theta\right)\Big|_0^{\pi/2}$

$4\left(\pi/2 - \cancel{\frac{1}{4}\sin 2\pi} - \cancel{(0 - 0)}\right) = \boxed{2\pi}$

3. (Calc Active) Interior of $r = 2 - \sin\theta$ on $[0, 2\pi]$



$2 - \sin\theta = 0$

$-\sin\theta = -2$

$\sin\theta = 2$

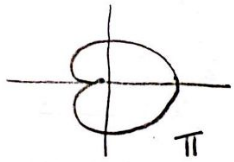
$\theta = \emptyset$

$\rightarrow$ go to period $[0, 2\pi]$

$A = \frac{1}{2} \int_0^{2\pi} (2 - \sin\theta)^2 \, d\theta$

$\boxed{14.137}$

4. (Calc Inactive) Interior of $r = 2 + 2\cos\theta$



$2 + 2\cos\theta = 0$

$\cos\theta = -1$

$\theta = \pi$

$A = 2 \cdot \frac{1}{2} \int_0^{\pi} (2 + 2\cos\theta)^2 \, d\theta$

$\int_0^{\pi} (4 + 8\cos\theta + 4\cos^2\theta) \, d\theta$

$= 4 \int_0^{\pi} (1 + 2\cos\theta + \cos^2\theta) \, d\theta$

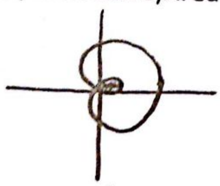
$= 4 \int_0^{\pi} \left(1 + 2\cos\theta + \frac{1}{2}(1 + \cos 2\theta)\right) d\theta$

$= 4 \int_0^{\pi} \left(1 + 2\cos\theta + \frac{1}{2} + \frac{1}{2}\cos 2\theta\right) d\theta$

$4\left[\theta + 2\sin\theta + \frac{1}{2}\theta + \frac{1}{4}\sin 2\theta \Big|_0^{\pi}\right]$

$4\left[\left(\pi + 0 + \frac{\pi}{2} + 0\right) - 0\right] = \boxed{6\pi}$

5. (Calc Active)Area of inner loop of $r = 1 + 2\cos\theta$



$1 + 2\cos\theta = 0$

$2\cos\theta = -1$

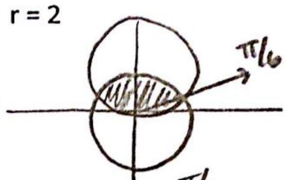
$\cos\theta = -\frac{1}{2}$

$\theta = \frac{2\pi}{3}, \frac{4\pi}{3}$

$$\frac{1}{2}\int_{2\pi/3}^{4\pi/3} (1 + 2\cos\theta)^2 \, d\theta$$

$.544$

6. (Calc Inactive) Common interior of $r = 4\sin\theta$ and r = 2



$4\sin\theta = 2$

$\sin\theta = \frac{1}{2}$

$\theta = \pi/6$

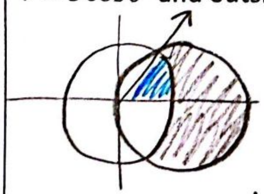
$$A = \cancel{2}\cdot\left[\cancel{\tfrac{1}{2}}\int_0^{\pi/6} (4\sin\theta)^2 d\theta + \cancel{\tfrac{1}{2}}\int_{\pi/6}^{\pi/2} 2^2 d\theta\right]$$

$$\int_0^{\pi/6} 16\sin^2\theta \, d\theta + \int_{\pi/6}^{\pi/2} 4 \, d\theta$$

$$16 \cdot \frac{1}{2}\int_0^{\pi/6} (1 - \cos 2\theta) d\theta + \int_{\pi/6}^{\pi/2} 4 \, d\theta$$

$$8\left[\theta - \tfrac{1}{2}\sin 2\theta\right]\Big|_0^{\pi/6} + 4\theta \Big|_{\pi/6}^{\pi/2} = \boxed{\frac{8\pi}{3} - 2\sqrt{3}}$$

7. (Calc Inactive - Only set up the integral) Inside $r = 3\cos\theta$ and outside $r = 2 - \cos\theta$



$3\cos\theta = 2 - \cos\theta$

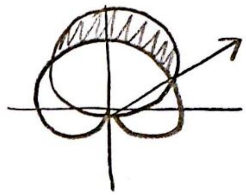
$4\cos\theta = 2$

$\cos\theta = \frac{1}{2}$

$\theta = \frac{\pi}{3}, \frac{5\pi}{3}$

$$A = 2\cdot\left[\frac{1}{2}\int_0^{\pi/3} (3\cos\theta)^2 d\theta - \frac{1}{2}\int_0^{\pi/3} (2 - \cos\theta)^2 d\theta\right]$$

8. (Calc Inactive - Only set up the integral) Inside $r = 3\sin\theta$ and outside $r = 1 + \sin\theta$



$3\sin\theta = 1 + \sin\theta$

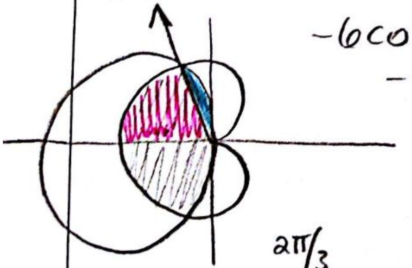
$2\sin\theta = 1$

$\sin\theta = 1/2$

$\theta = \pi/6, 5\pi/6$

$$2\cdot\left[\frac{1}{2}\int_{\pi/6}^{\pi/2} (3\sin\theta)^2 d\theta - \int_{\pi/6}^{\pi/2} (1 + \sin\theta)^2 d\theta\right.$$

9. (Calc Inactive - Only set up the integral) Area common to $r = -6\cos\theta$ and $r = 2 - 2\cos\theta$



$-6\cos\theta = 2 - 2\cos\theta$

$-4\cos\theta = 2$

$\cos\theta = -\frac{1}{2}$

$\theta = \frac{2\pi}{3}, \frac{4\pi}{3}$

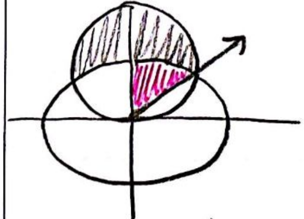
$$2\cdot\frac{1}{2}\int_{\pi/2}^{2\pi/3} (-6\cos\theta)^2 d\theta = 1.631$$

$+$

$$2\cdot\frac{1}{2}\int_{2\pi/3}^{\pi} (2 - 2\cos\theta)^2 d\theta = 14.077$$

$= 15.708$

10. (Calc Inactive - Only set up the integral) Inside $r = 3\sin\theta$ and outside $r = 2 - \sin\theta$



$3\sin\theta = 2 - \sin\theta$

$4\sin\theta = 2$

$\sin\theta = \frac{1}{2}$

$\theta = \pi/6$

$$2\cdot\left[\frac{1}{2}\int_{\pi/6}^{\pi/2} (3\sin\theta)^2 d\theta - \frac{1}{2}\int_{\pi/6}^{\pi/2} (2 - \sin\theta)^2 d\theta\right]$$

$\vdots$

$3\sqrt{3}$