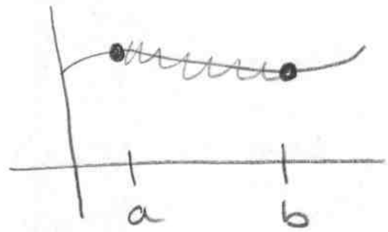
Unit 4 Lesson 1:

10.2 Parametric Equations and Length of Curve

* Formula for Length of Curve:

$$S = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$$



1) Find length of curve $y = \ln(\cos x)$ from $[0, \frac{\pi}{3}]$

$$S = \int_0^{\pi/3} \sqrt{1 + (-\tan x)^2}\, dx = 1.317$$

$$y' = \frac{1}{\cos x} \cdot -\sin x = -\tan x$$

YT

2) Find length of curve $x = y\sin y + \cos y$ from $\underbrace{[0, \pi]}_{y\text{-values}}$

$$S = \int_a^b \sqrt{1 + [g'(y)]^2}\, dy$$

$$\frac{dx}{dy} = y\cos y + \cancel{\sin y} - \cancel{\sin y}$$

$$S = \int_0^{\pi} \sqrt{1 + (y\cos y)^2}\, dy = 4.8478$$

* Parametric Equations:

* Let $x = f(t)$ and $y = g(t)$ where f and g are two functions whose common domain is some interval I.

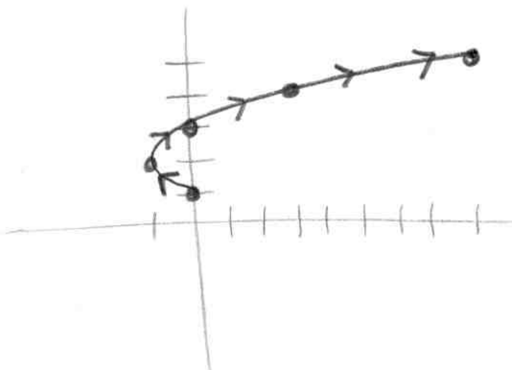
The equations

$$x = f(t) \quad y = g(t)$$

are parametric equations of the curve

(t is a parameter)

1) Graph $x = t^2 - 2t$
$y = t + 1$
for $0 \le t \le 4$



t	x	y
0	0	1
1	-1	2
2	0	3
3	3	4
4	8	5

Eliminate the parameter:

$$t = y - 1$$

$$x = (y-1)^2 - 2(y-1)$$

$$x = y^2 - 2y + 1 - 2y + 2$$

$$x = y^2 - 4y + 3$$

Eliminate parameter.

2) $x = t - 2$ $\qquad t = x + 2$

YT $y = 1 - \sqrt{t}$

$y = 1 - \sqrt{x+2}$

3) $x = \cos\theta$ $\qquad x^2 = \cos^2\theta$

$y = \sin\theta$ $\qquad y^2 = \sin^2\theta$

$x^2 + y^2 = \cos^2\theta + \sin^2\theta$

$x^2 + y^2 = 1$

4) $x = \cos\theta \longrightarrow x^2 = \cos^2\theta$

YT $y = \cos 2\theta$

$\cos 2\theta = 2\cos^2\theta - 1$

$y = 2x^2 - 1$

MORE:

1) Length of Curve: $f(x) = \frac{4\sqrt{2}}{3} x^{3/2} \quad 0 \le x \le 1$

$f'(x) = \frac{3}{2}\left(\frac{4\sqrt{2}}{3}\right) x^{1/2} = 2\sqrt{2}\, x^{1/2}$

$S = \int_0^1 \sqrt{1 + (2\sqrt{2}x^{1/2})^2}\, dx = \int_0^1 \sqrt{1 + 8x}\, dx$

$= \frac{13}{6}$

2) Length of Curve: $f(x) = \frac{x^3}{6} + \frac{1}{2x}$ $[\frac{1}{2}, 2]$

$f'(x) = \frac{3}{6}x^2 - \frac{1}{2}x^{-2}$

$f'(x) = \frac{x^2}{2} - \frac{1}{2x^2}$

$S = \int_{1/2}^{2} \sqrt{1 - (\frac{x^2}{2} - \frac{1}{2x^2})^2}\, dx = 2.0625$

or $\frac{33}{16}$

3) Length of Curve: $(y-1)^3 = x^2$ $[0, 8]$

$[(y-1)^3]^{1/3} = [x^2]^{1/3}$

$y - 1 = x^{2/3}$

$y = x^{2/3} + 1$ $\quad y' = \frac{2}{3}x^{-1/3}$

$S = \int_0^8 \sqrt{1 + (\frac{2}{3}x^{-1/3})^2}\, dx \approx 9.073$

4) Eliminate parameter:

$x = t^2 - 4$

$y = \frac{t}{2}$ $\quad 2y = t$

$x = (2y)^2 - 4$

$x = 4y^2 - 4$

5) Eliminate parameter:

$x = 3\cos\theta$ $\quad \frac{x}{3} = \cos\theta$

$y = 4\sin\theta$ $\quad \frac{y}{4} = \sin\theta$

$\sin^2\theta + \cos^2\theta = 1$

$(\frac{y}{4})^2 + (\frac{x}{3})^2 = 1$

$\frac{y^2}{16} + \frac{x^2}{9} = 1$

(ellipse)

10.3 Parametric Equations + Calculus

$$x = f(t)$$
$$y = g(t)$$

$$\text{Slope} = \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

2nd Derivative:

$$\frac{d^2y}{dx^2} = \frac{d(y')}{dx} = \frac{\frac{d(y')}{dt}}{\frac{dx}{dt}}$$

1) Given $x = \sin t$ and $y = \cos t$
Find $\frac{dy}{dx}$ when $t = \pi/4$. Write eqn of tangent line when $t = \frac{\pi}{4}$.

$$\frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-\sin t}{\cos t} = -\tan t \quad @\ t = \frac{\pi}{4} = -1$$

point: $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ slope $= -1$

$$y - \frac{\sqrt{2}}{2} = -1\left(x - \frac{\sqrt{2}}{2}\right)$$

2) $x = \sqrt{t}$

YT $y = \frac{1}{4}(t^2 - 4)$ for $t \geq 0$

$= \frac{1}{4}t^2 - 1$

Find slope and concavity at $(2, 3)$.

slope: $\dfrac{\frac{dy}{dt}}{\frac{dx}{dt}} = \dfrac{\frac{1}{2}t}{\frac{1}{2}t^{-1/2}} = t^{3/2}$

$2 = \sqrt{t}$

$4 = t$

@ $t = 4$

slope $= 8$

concavity: $\dfrac{\left(\frac{dy}{dx}\right)'}{\frac{dx}{dt}} = \dfrac{\frac{3}{2}t^{1/2}}{\frac{1}{2}t^{-1/2}} = 3t$ @ $t = 4$

$= 12$

$\therefore$ Concave up

3) $x = \cos\theta$

$y = 3\sin\theta$

Find slope and concavity when $\theta = 0$

slope: $\dfrac{\frac{dy}{dt}}{\frac{dx}{dt}} = \dfrac{3\cos\theta}{-\sin\theta} = -3\cot\theta$ @ $\theta = 0$

slope = undefined

(vertical tangent)

concavity: $\dfrac{3\csc^2\theta}{-\sin\theta} = -3\csc^3\theta$ @ $\theta = 0$

undefined

(possible P.O.I.)

neither cc↑ or ↓

* Curve Length (Parametric Form):

$$s = \int_a^b \sqrt{\left[\frac{dx}{dt}\right]^2 + \left[\frac{dy}{dt}\right]^2}\, dt$$

$$S = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}\, dt$$

1) Given $x = 5\cos t - \cos 5t$
$y = 5\sin t - \sin 5t$

Find length of curve over $[0, \pi/2]$

$$S = \int_0^{\pi/2} \sqrt{(-5\sin t + 5\sin 5t)^2 + (5\cos t - 5\cos 5t)^2}\, dt$$
$$= \boxed{10}$$

$$\frac{dx}{dt} = -5\sin t + 5\sin 5t$$

$$\frac{dy}{dt} = 5\cos t - 5\cos 5t$$

MORE

1) $x = \theta - \sin\theta$ Find y' and y''
$y = 1 - \cos\theta$

$$y' = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\sin\theta}{1 - \cos\theta}$$

$$y'' = \frac{\frac{d(y')}{d\theta}}{\frac{dx}{d\theta}} = \frac{\frac{(1-\cos\theta)\cos\theta - \sin\theta(\sin\theta)}{(1-\cos\theta)^2}}{(1-\cos\theta)}$$

$$= \frac{\cos\theta - \cos^2\theta - \sin^2\theta}{(1-\cos\theta)^3} = \frac{\cos\theta - (\cos^2\theta + \sin^2\theta)}{(1-\cos\theta)^3}$$

$$= \frac{\cos\theta - 1}{(1-\cos\theta)^3} = \frac{-1(1-\cos\theta)}{(1-\cos\theta)^3} = \frac{-1}{(1-\cos\theta)^2}$$

2) $x = \sqrt{t+6}$ $(t+6)^{1/2}$ Find y' and y''
$y = \sqrt{6t} = (6t)^{1/2}$

$$y' = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\frac{3}{\sqrt{6t}}}{\frac{1}{2\sqrt{t+6}}} = \frac{3}{\sqrt{6t}} \cdot \frac{2\sqrt{t+6}}{1} = \frac{6\sqrt{t+6}}{\sqrt{6t}}$$

$$y'' = \frac{\frac{d\left(\frac{6\sqrt{t+6}}{\sqrt{6t}}\right)}{dt}}{\frac{dx}{dt}} = \frac{\frac{\sqrt{6t} \cdot \frac{6}{2\sqrt{t+6}} - 6\sqrt{t+6} \cdot \frac{1}{2\sqrt{6t}} \cdot 6}{(\sqrt{6t})^2}}{\frac{1}{2\sqrt{t+6}}}$$

$$= \frac{\frac{3\sqrt{6t}}{\sqrt{t+6}} \cdot \frac{-18\sqrt{t+6}}{\sqrt{6t}}}{6t} \cdot 2\sqrt{t+6}$$

3) Equation of tangent line at $t=-1$

$x=-1 \quad x=\frac{1}{t}$

$y=-1 \quad y=t$

$$\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{1}{\frac{-1}{t^2}}=-t^2$$
$$=-1$$

$y-y_1=m(x-x_1)$

$y+1=-1(x+1)$

4) Find $\frac{dy}{dx}$ and horizontal and vertical tangents.

$x=t^2-t+1$

$y=t^3-3t$

$$\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{3t^2-3}{2t-1}$$

$3t^2-3=0$

$3t^2=3$

$t^2=1$

$t=\pm 1$

$t=1 \quad (1,-2)$

$t=-1 \quad (3,2)$

$2t-1=0$

$2t=1$

$t=\frac{1}{2}$

$(\frac{3}{4}, \frac{-11}{8})$

$\frac{1}{4}-\frac{1}{2}+1$

$\frac{1}{8}-\frac{3}{2}$

~~5)~~ Find $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, and slope/concavity at $\theta = 0$

$x = \cos\theta$
$y = 3\sin\theta$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{3\cos\theta}{-\sin\theta} = \quad y'(0) = \text{undefined}$$

$-3\cot\theta$

$$\frac{d^2y}{dx^2} = \frac{3\csc^2\theta}{-\sin\theta} = -3\csc^3\theta$$

$y''(0) = \text{undefined}$

6) Length of Curve:

$0 \le t \le \pi$

$x = e^t \cos t$
$y = e^t \sin t$

$$\int_0^{\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$\frac{dx}{dt} = e^t(-\sin t) + \cos t e^t = e^t(-\sin t + \cos t)$

$\frac{dy}{dt} = e^t(\cos t) + \sin t(e^t) = e^t(\cos t + \sin t)$

$$\int_0^{\pi} \sqrt{[e^t(-\sin t + \cos t)]^2 + [e^t(\sin t + \cos t)]}\, dt$$

$=$